

## Example: $LU$ Factorization with Partial Pivoting (Numerical Linear Algebra, Spring 2016)

**Problem:**

Given  $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix}$ , use partial pivoting to find the  $LU$  decomposition  $PA = LU$

where  $P$  is the associated permutation matrix.

**Solution:** We can keep the information about permuted rows of  $A$  in the permutation vector  $\mathbf{p} = (1, 2, 3)^T$  which initially shows the original order of the rows. Recall that we find the largest entry in the column in absolute value and use it as the pivot element: we multiply  $A$  by the matrix  $P_1$  from the left. The permutation vector becomes  $\mathbf{p} = (3, 2, 1)^T$  and

$$P_1 A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 8 & 0 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}.$$

Next, we do the first column elimination by left-multiplication of  $L_1$ :

$$L_1 P_1 A = \begin{pmatrix} 1 & 0 & 0 \\ -4/7 & 1 & 0 \\ -1/7 & 0 & 1 \end{pmatrix} \begin{pmatrix} 7 & 8 & 0 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 8 & 0 \\ 0 & 3/7 & 6 \\ 0 & 6/7 & 3 \end{pmatrix}.$$

Find the second pivot by left-multiplication of  $P_2$ . The permutation vector becomes  $\mathbf{p} = (3, 1, 2)^T$  and

$$P_2 L_1 P_1 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 7 & 8 & 0 \\ 0 & 3/7 & 6 \\ 0 & 6/7 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 8 & 0 \\ 0 & 6/7 & 3 \\ 0 & 3/7 & 6 \end{pmatrix}.$$

Apply the second column elimination the left-multiplication of  $L_2$ :

$$L_2 P_2 L_1 P_1 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{pmatrix} \begin{pmatrix} 7 & 8 & 0 \\ 0 & 6/7 & 3 \\ 0 & 3/7 & 6 \end{pmatrix} = \begin{pmatrix} 7 & 8 & 0 \\ 0 & 6/7 & 3 \\ 0 & 0 & 9/2 \end{pmatrix}.$$

Therefore,  $PA = LU$  where

$$L = L_1^{-1}L_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4/7 & 1 & 0 \\ 1/7 & 1/2 & 1 \end{pmatrix}$$

with  $L_1 = P_2L_1P_2^{-1}$ ,

$$U = \begin{pmatrix} 7 & 8 & 0 \\ 0 & 6/7 & 3 \\ 0 & 0 & 9/2 \end{pmatrix}$$

and

$$P = P_2P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$