Example: LU Factorization with Partial Pivoting (Numerical Linear Algebra, MTH 365/465)

Given $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix}$, use Gaussian elimination with partial pivoting to find the LU decomposition $PA = LU$ where P is the associated permutation matrix.

Solution: We can keep the information about permuted rows of A in the permutaion vector $\mathbf{p} = (1, 2, 3)^T$ which initially shows the original order of the rows. Recall that we find the largest entry in the column in absolute value and use it as the pivot element: we multiply A by the matrix P_1 from the left. The permutation vector becomes $\mathbf{p} = (3, 2, 1)^T$ and

$$P_1 A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 8 & 0 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}.$$

Next, we do the first column elimination by left-multiplication of L_1:

$$L_1 P_1 A = \begin{pmatrix} 1 & 0 & 0 \\ -4/7 & 1 & 0 \\ -1/7 & 0 & 1 \end{pmatrix} \begin{pmatrix} 7 & 8 & 0 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 8 & 0 \\ 0 & 3/7 & 6 \\ 0 & 6/7 & 3 \end{pmatrix}.$$

Find the second pivot by left-multiplication of P_2. The permutation vector becomes $\mathbf{p} = (3, 1, 2)^T$ and

$$P_2 L_1 P_1 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 7 & 8 & 0 \\ 0 & 3/7 & 6 \\ 0 & 6/7 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 8 & 0 \\ 0 & 6/7 & 3 \\ 0 & 3/7 & 6 \end{pmatrix}.$$

Apply the second column elimination by left-multiplication of L_2:

$$L_2 P_2 L_1 P_1 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{pmatrix} \begin{pmatrix} 7 & 8 & 0 \\ 0 & 6/7 & 3 \\ 0 & 3/7 & 6 \end{pmatrix} = \begin{pmatrix} 7 & 8 & 0 \\ 0 & 6/7 & 3 \\ 0 & 0 & 9/2 \end{pmatrix}.$$

Therefore, $PA = LU$ where

$$L = L_1'^{-1} L_2'^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4/7 & 1 & 0 \\ 1/7 & 1/2 & 1 \end{pmatrix}$$

with $L_1' = P_2 L_1 P_2^{-1}$, and $L_2' = L_2$,

$$U = \begin{pmatrix} 7 & 8 & 0 \\ 0 & 6/7 & 3 \\ 0 & 0 & 9/2 \end{pmatrix},$$

and

$$P = P_2 P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In general, for an $n \times n$ matrix A, the LU factorization provided by Gaussian elimination with partial pivoting can be written in the form:

$$(L_{n-1}' \cdots L_2' L_1')(P_{n-1} \cdots P_2 P_1) A = U,$$

where $L_i' = P_{n-1} \cdots P_{i+1} L_i P_{i+1}^{-1} \cdots P_{n-1}^{-1}$.
If $L = (L_{n-1}' \cdots L_2' L_1')^{-1}$ and $P = P_{n-1} \cdots P_2 P_1$, then $PA = LU$.