Summary on Square Matrices

An $n \times n$ matrix A is nonsingular if we can solve $A\boldsymbol{x} = \boldsymbol{b}$ for any $\boldsymbol{b} \in \mathbb{R}$. Otherwise, A is singular.

The following statements are equivalent:

- A is nonsingular.
- A is invertible (A^{-1} exists).
- A row echelon form of A has only nonzero entries along the diagonal (in Gaussian elimination with back substitution).
- System $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of n-vector $\mathbf{b}$.
- System $A\mathbf{x} = \mathbf{0}$ has the trivial solution only.
- The columns of A are linearly independent.
- $\det(A) \neq 0$ (and $\det(A^{-1}) = 1/\det(A)$).

From above, A is singular $\Leftrightarrow \det(A) = 0 \Leftrightarrow A\mathbf{x} = \mathbf{0}$ has a nonzero solution.