MTH/CS 364/464 Numerical Analysis, Fall 2016
FINAL EXAM
Due at 10am on Saturday, 12/17/16

Instructions: This is a take-home exam. You may *not* discuss the exam problems with anyone but me, the work should be yours only. Please write your solutions on your own paper, show your work. Start a new page for each problem. For the programming exercises also hand in your Matlab codes and outputs. Your graphs should be annotated and clear to read.

1. (14pts) Find the *natural* cubic spline S that interpolates the data $f(0) = 0$, $f(1) = 2$, $f(2) = 1$, $f(3) = 0$. Graph the spline with the data points (generate your graph in MATLAB).

2. (12pts) Determine constants a, b, c, and d that will produce a quadrature formula

$$\int_{-1}^{1} f(x)dx = af(-1) + bf(1) + cf'(-1) + df'(1)$$

   that has degree of precision three.

3. (20pts) Consider the composite *midpoint rule* for approximating an integral

$$\int_a^b f(x)dx \approx h\Sigma_{i=1}^n f\left(\frac{x_i + x_{i-1}}{2}\right),$$

   where $h = (b-a)/n$ and $x_i = a + ih$, $i = 0, 1, \ldots, n$.

   (a) Generate a plot in MATLAB to illustrate the midpoint rule geometrically, i.e., show what area is being computed by the formula (use any function f).

   (b) Show this formula is exact if f is either constant or linear in each subinterval.

   (c) Assuming that $f \in C^2[a, b]$, show that the midpoint rule is second-order accurate; that is, the error is less than or equal to a constant times h^2. To do this, you will first need to show that the error in each subinterval is of order h^3. To see this, expand f in a Taylor series about the midpoint $x_{i-1/2} = (x_i + x_{i-1})/2$ of the subinterval:

$$f(x) = f(x_{i-1/2}) + (x - x_{i-1/2})f'(x_{i-1/2}) + \frac{(x - x_{i-1/2})^2}{2} f''(\xi_{i-1/2}),\ \xi_{i-1/2} \in [x_{i-1}, x_i].$$

   By integrating each term, show that the difference between the true value $\int_{x_{i-1}}^{x_i} f(x)\,dx$ and the approximation $hf(x_{i-1/2})$ is of order h^3. Finally, combine the results from all subintervals to show that the total error is of order h^2.

4. (12pts) Write down the result of applying one step of Euler's method to the initial value problem $y' = (t+1)e^{-y}$, $y(0) = 0$, using step size $h = 0.1$. Do the same for the midpoint method and for Heun's method.

5. (12pts) Write down the result of applying one step of Euler's method to the predator-prey equations (R - rabbits, F - foxes):

$$\begin{aligned} R' &= (2-F)R, \\ F' &= (R-2)F, \end{aligned}$$

starting with $R_0 = 2$ and $F_0 = 1$ and using stepsize $h = 0.1$. Do the same for the midpoint method and for Heun's method.

6. (15pts) Continuing the love saga from Section 11.2.6, Juliet's emotional swings lead to many sleepless nights, which consequently dampens her emotions. Mathematically, the pair's love can now be expressed as

$$\begin{aligned} dx/dt &= -0.2y, \\ dy/dt &= 0.8x - 0.1y, \end{aligned}$$

Suppose this state of the romance begins when Romeo is smitten with Juliet ($x(0) = 2$) and Juliet is indifferent ($y(0) = 0$).

   (a) Explain how these equations reflect Juliet's dampened emotions (compare to equations (11.23) in the textbook).

   (b) Use *ode45* to produce three graphs in MATLAB, like those in Figure 11.7 in the textbook, showing Romeo and Juliet's love for $0 \le t \le 60$.

   (c) From your graphs, describe how the change in Juliet described in this exercise will affect the relationship and its eventual outcome?

7. (15pts) Consider the initial value problem

$$y' = \frac{-ty}{\sqrt{2-y^2}}, \quad 0 \le t \le 5, \ y(0) = 1.$$

   (a) Write your own MATLAB routine to approximate the solution using the classical fourth-order Runge-Kutta method with $h = 0.25$. Graph the approximations.

   (b) Solve the IVP using *ode45*. Graph the solution found.