Numerical Linear Algebra, Spring 2018
FINAL EXAM
Due by 10am on Friday May 11th.

NAME: ______________________________ **SCORE:** ________

Instructions: This is a take-home exam. I expect your solutions to be well-written, neat, and organized. Feel free to type up your final version using LaTeX. The rules for the exam are the following:

1. You should turn in this cover page and all of the work that you have decided to submit. Please write your solutions on your own paper, show your work. Start a new page for each problem.
2. For MATLAB results, turn in all your codes and results in a proper order. Make sure to comment your codes, annotate your graphs, and put all the explanations, codes, graphs, and results **together** for each problem.

   Poorly organized solutions may result in losing points.
3. You may use your textbook, lecture notes, homework and midterm exam solutions, but you are NOT allowed to consult any other sources, when working on the exam. This includes your classmates, people outside of the class, and online resources.
4. You are NOT allowed to copy someone else's work or let someone else copy your work.

Attention:
MTH 365 students are required to do problems 1-5;
MTH 465 students are required to do all the problems.

To convince me that you have read and understand the instructions, sign in the box below:

SIGNATURE:

Good luck and have fun!

1. (22pts) Write a MATLAB code to implement the *inverse iteration with shift* for the matrix $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 5 & 1 \\ -2 & -1 & 9 \end{pmatrix}$. Use your routine *to compute all the eigenvalues and corresponding eigenvectors of* A by choosing an appropriate shift s each time. Explain how you choose your shifts. Submit your code and output. Check your results by computing eigenvalues directly.

2. (18pts) Consider the two-point boundary value problem

$$u'' + 2xu' - x^2 u = x^2, \; u(0) = 1, \; u(1) = 0.$$

   (a) Let $h = 1/4$ and explicitly write out the difference equations, using centered-difference quotients for all derivatives.

   (b) Convert the equations to a linear system $A\mathbf{x} = \mathbf{b}$, and then solve the system by hand or using MATLAB.

3. (16pts) Answer the following questions:

   (a) Show that if a square matrix A has decomposition $A = LL^T$ with L nonsingular, then A is symmetric positive definite.

   (b) Find (by hand) the Cholesky factor L in the decomposition $A = LL^T$ for the matrix $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 8 & 0 \\ 2 & 0 & 24 \end{pmatrix}$.

4. (8pts) Let A be a positive definite $n \times n$ matrix. For any $\mathbf{x} \in \mathbb{R}^n$, let $||\mathbf{x}||_A = \sqrt{\langle A\mathbf{x}, \mathbf{x} \rangle}$. Show that this defines a vector norm on $\mathbb{R}^n$.

5. (24pts) Given the data below, and using MATLAB,

x	4.0	4.2	4.5	4.7	5.1	5.5	5.9	6.3	6.8	7.1
y	102.56	113.18	130.11	142.05	167.53	195.14	224.87	256.73	299.50	326.72

   (a) construct the least squares polynomial of degree 1 and compute the error;

   (b) construct the least squares polynomial of degree 3 and compute the error;

   (c) construct the least squares approximation of the form $y = be^{ax}$ and compute the error (*hint: take logarithm on both sides*).

For each polynomial, provide its graph together with the data points from the table. Submit all your codes and results.

6. (12pts) Assume $A \in \mathbb{C}^{n\times n}$ and $\exists\, p \geq 1$, s.t. $\|A\|_p < 1$, where $||.||_p$ is a vector-induced matrix norm.

   (a) Prove that $I - A$ is invertible.

   (b) Assuming that the series $\sum_{k=0}^{\infty} A^k$ converges, prove that $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$.