

## WRITING MATHEMATICS

In any discipline, the successful communication of ideas is at least as important as the ideas themselves. Most disciplines develop standard usages and restrictions that differ from everyday English. Mathematics is not an exception. This handout is intended to alert you to some (but certainly not all) of the peculiarities of writing mathematics.

### GROUND RULES

There are two components of writing mathematics: mathematics (content) and writing (form). Each is important and neither can be ignored.

- Every statement you make must be mathematically correct. If you are not sure of the validity of a claim, first convince yourself. Then include in your write-up the details you required in order to reach a conclusion. It is better to make no assertion than to make a false assertion.

– **Example.** If  $x > 0$ , then  $x^2 \leq 2^x$ .

This statement sounds nice and the inequality holds for  $x = 1$  and  $x = 2$ , so it is tempting just to believe it always holds. Unfortunately, it doesn't. To convince ourselves of that, all we need to do is to show a single *counterexample*, an example for which the 'if' part holds but the 'then' part does not. Try  $x = 3$ . Certainly  $3 > 0$ , so the 'if' part holds. However,  $3^2 = 9$  and  $2^3 = 8$ , so when  $x = 3$ ,  $x^2 > 2^x$ . This counterexample shows us that the original statement is false.

– **Example.**  $\sin\left(\frac{5001\pi}{2}\right) = 1$ .

This statement is true, but it is certainly not obvious. The author should include a simple explanation or a few details to convince the reader. For example, the author may write,

Since  $\sin(x)$  is  $2\pi$ -periodic, *i.e.*,  $\sin(x + 2n\pi) = \sin(x)$  for any integer  $n$ , it follows that

$$\sin\left(\frac{5001\pi}{2}\right) = \sin\left(\frac{\pi}{2} + 1250(2\pi)\right) = \sin\left(\frac{\pi}{2}\right) = 1.$$

– **Example.** Since  $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$  is continuous,  $f$  is integrable.

While it is true that every continuous function is integrable, the function given above is **not** continuous, so we cannot conclude that it is integrable. (We also cannot conclude that  $f$  is *not* integrable.)

- **Example.** Since  $|x|$  is continuous, it is differentiable.

In this case, the function mentioned is indeed continuous. However, it is not true that a continuous function must be differentiable, so we cannot conclude that  $|x|$  is differentiable. (We also cannot conclude that it is *not* differentiable.)

- The rules of standard spelling and grammar remain in effect when writing mathematics. In particular, you must use complete sentences and well-structured paragraphs.

- **Example.** Since  $f$  is a function.

What happens since  $f$  is a function? The author has left the reader in suspense with this incomplete sentence.

- **Example.** Suppose  $x =$ .

This is not a complete sentence. The verb ‘=’ is transitive and requires an object.

- **Example.**  $x^2 + 2x + 5$

This is not a complete sentence; it contains no verb. Disjointed phrases such as this should be eliminated from your work.

- **Example.**  $(a + b)^2 = a^2 + 2ab + b^2$ .

This is a perfectly valid complete sentence. It has subject ‘ $(a + b)^2$ ’, transitive verb ‘=’ and object ‘ $a^2 + 2ab + b^2$ .’ Many other mathematical symbols may also serve as verbs (e.g.,  $>$ ,  $\geq$ ,  $<$ ,  $\leq$ ,  $\neq$ ,  $\in$ ).

## RESTRICTIONS ON WORD AND SYMBOL USAGE

In everyday English, symbols are used infrequently. In mathematics, they are used regularly as an integral facet of communication. As a result, mathematics places additional restrictions on the use of certain symbols.

- Symbols that have a specific mathematical meaning are reserved for mathematical use.
  - **Example.** Let  $a + b$  be positive. Then the product  $ab$  is also positive.

While the author may have intended for  $a$  **and**  $b$  each to be greater than zero, he has instead indicated that the quantity  $(a + b)$  is greater than zero. The misstatement allows, among others, the case  $a = 0$  and  $b = 1$  since then  $a + b = 1$ , which is indeed positive. In this case, however, the product  $ab$  is 0, which is not positive. The author should replace the symbol ‘+’ by the written word ‘and.’

- Always respect the ‘equals’ sign.

– **Example.** Let  $f(x) = x^2 = \frac{d}{dx}x^2 = 2x$ .

This chain of “equalities” is false; only for the values  $x = 0$  and  $x = 2$  do we have  $x^2 = 2x$ . The problem entered when the author set the function equal to its own derivative. One way to fix this would be to say:

$$\text{Let } f(x) = x^2. \text{ Then } \frac{d}{dx}(f(x)) = \frac{d}{dx}(x^2) = 2x.$$

– **Example.**  $\lim_{x \rightarrow \infty} \frac{x}{x^2} = \frac{1}{x} = 0$ .

The first “equality” is incorrect; the limit as  $x$  approaches infinity of the function  $\frac{x}{x^2}$  is not the function  $\frac{1}{x}$ . Since one step in the chain of “equalities” is invalid, the entire chain fails. The following statement, however, **is** true:

$$\lim_{x \rightarrow \infty} \frac{x}{x^2} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

- Mathematics is case-sensitive.

– **Example.** If  $a = 3$ , then  $A + 4 = 7$ .

The author here has changed  $a$  to  $A$ . While  $a$  has been defined,  $A$  has not, so the implication does not necessarily hold. Do not use upper-case and lower-case versions of the same letter interchangeably unless you specifically define them to represent the same quantity.

- More generally, define any terms you use.

– **Example.** The area of a right circular cylinder is  $2\pi rh$ .

There are two problems here. The more obvious is that the author has not specified what  $r$  and  $h$  denote. A less obvious problem is the meaning of ‘area’. Does the author intend to include or exclude the area of the top and bottom? A more precise (and hence better) phrasing is

“The lateral surface area of a right circular cylinder is  $2\pi rh$ , where  $r$  is the radius and  $h$  is the height of the cylinder.”

or

“The surface area, excluding the top and bottom, of a right circular cylinder is  $2\pi rh$ , where  $r$  is the radius and  $h$  is the height of the cylinder.”

- **Example.** Let  $f(x) = e^x$ . Then  $F(x) = e^x$ .

What is  $F$ ? How is it related to the given function,  $f$ ? Is  $F$  the derivative of  $f$ ? The anti-derivative? Something else? The author must let the reader know how the conclusion that  $F(x) = e^x$  was reached.

- Once you’ve assigned a variable name, don’t re-use it for a different meaning within the same context.
  - **Example.** Suppose  $f$  and  $g$  are each linear functions. Then  $f(x) = ax + b$  and  $g(x) = ax + b$  for some real numbers  $a$  and  $b$ .

The last sentence is only true when  $f$  and  $g$  are the exact same function, which is much more restrictive than the original assumption. Once you have assigned roles to  $a$  and  $b$ , they are no longer free to take on other assignments. Use some letter other than  $f, g, a$  or  $b$  here. For example, say, “ $g(x) = cx + d$  for some real numbers  $c$  and  $d$ .”

## PRECISION OF PHRASING

Standard English allows for a degree of vagueness that is generally unacceptable in the communication of mathematical ideas. Mathematics is a very precise discipline, and it is best to replace vague phrasing with the appropriate specific terminology.

- There is a distinction between the definite article (‘the’) and the indefinite articles (‘a’, ‘an’). ‘The’ implies that there is only one such object.
  - **Example.** The antiderivative of  $x^3$  is  $\frac{1}{4}x^4$ .

While it is certainly the case that  $\frac{d}{dx}(\frac{1}{4}x^4) = x^3$ , it is also the case that  $\frac{d}{dx}(\frac{1}{4x^4} + 1) = x^3$ . Since  $\frac{1}{4}x^4$  is not the *only* antiderivative of  $x^3$ , using ‘the’ above is misleading. Instead, say, “*An* antiderivative of  $x^3$  is  $\frac{1}{4}x^4$ .”

- **Example.** A simultaneous solution of the system of equations  $\left\{ \begin{array}{l} 3x + y = 5 \\ x - y = 3 \end{array} \right\}$  is  $x = 2$  and  $y = -1$ .

This is a correct statement. However, since the given solution is in fact the *unique* simultaneous solution to the system of equations, it is much more informative to tell the reader, “*The (unique)* simultaneous solution . . . is  $x = 2$  and  $y = -1$ .”

- Avoid the use of imprecise terms.

- **Example.**  $x^2 - 4$  has a real root since 2 makes it 0.

The sentence is unclear; the verb ‘makes’ and the object ‘it’ are both vague. The author should be more specific. For example, it would be preferable to say:

The function  $x^2 - 4$  has a real root; indeed, when  $x = 2$ ,  
 $x^2 - 4 = 2^2 - 4 = 0$ .

- **Example.** Suppose  $y$  is a real number between 3 and 5.

Is the author here *including* or *excluding* the possibilities that  $y = 3$  and/or  $y = 5$ ? The word ‘between’ is somewhat nebulous. For clarity, it is best to modify ‘between’ as appropriate:

$y$  is *strictly* between 3 and 5

or

$y$  is between 3 and 5 *inclusive*.

Alternatively, the author could avoid the use of ‘between’ entirely by writing

$y \in (3, 5)$  (or  $y \in [3, 5]$  as appropriate)

or by writing

$3 < y < 5$  (or  $3 \leq y \leq 5$  as appropriate).

- **Example.** Let  $x$  and  $y$  be real numbers. Then either  $x > y$  or  $y > x$ .

The author has not excluded the possibility that  $x = y$ , in which case his conclusion does not hold. If the author intends to exclude the case  $x = y$  he may say:

Let  $x$  and  $y$  be two *distinct* real numbers.

Otherwise, he should modify the second sentence to read, for example:

Then either  $x > y$  or  $y \geq x$ .

- Make certain that your use of a term agrees with the definition of that term.

- **Example.** If  $a$  is a non-negative integer then  $-a$  is a negative integer.

The definition of non-negative is *not less than zero*. In particular, 0 is a non-negative integer, so the implication is false since  $(-0) = 0$ , which is not negative. The author should replace ‘non-negative’ with ‘positive.’

- **Example.**  $\arcsin(\sin(\frac{7\pi}{4})) = \frac{7\pi}{4}$ .

In order to be one-to-one, we define the inverse sine function to yield only values in the range  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ , so  $\frac{7\pi}{4}$  is an impossible answer. The correct statement is

$$\arcsin\left(\sin\left(\frac{7\pi}{4}\right)\right) = -\frac{\pi}{4}.$$

- **Example.** Let  $A = \{1, 2, 3\}$ . Then  $3 \subseteq A$ .

3 is an element of the set  $A$ , not a subset of  $A$ . The appropriate statement is either

$$3 \in A$$

or

$$\{3\} \subseteq A.$$

- **Example.** If  $x \in (-\infty, 1) \cup (3, \infty)$ , then  $1 > x > 3$ .

Any chain of inequalities contains hidden ‘and’s; ‘ $1 > x > 3$ ’ means ‘ $1 > x$  and  $x > 3$ .’ But if that is the case, then  $1 > 3$ , which is false. What the author intended was ‘ $1 > x$  or  $x > 3$ .’ There is no notational shorthand for this. If you use a chain of inequalities, make sure that the entire chain can be joined using the word ‘and’, with no ‘or’s. A single ‘or’ invalidates the whole chain.

- **Example.**  $\{x|x^2 = 4\} = \{x|x = \sqrt{4}\}$

These two sets are not equal.  $\{x|x^2 = 4\}$  is the set  $\{-2, 2\}$  while  $\{x|x = \sqrt{4}\}$  consists of the single element 2. The symbol ‘ $\sqrt{\quad}$ ’ is defined to be only the positive square root.

- Avoid using the same word to refer to different items without clarification.

- **Example.** Let  $f(x) = \sqrt{x+1}$  and let  $g(x) = \frac{1}{x}$ . The domain is  $\{x|x \neq 0\}$  and the other domain is  $\{x|x \geq -1\}$ , so one domain is  $\{x|x \leq -1 \text{ or } x > 0\}$  and the other domain is  $\{x|x > -1\}$ .

The author here has used the word ‘domain’ to refer to several different items. The result is confusing. One improvement might be:

Let  $f(x) = \sqrt{x}$  and let  $g(x) = \frac{1}{x}$ . The domain of  $f$  is  $\{x|x \geq 0\}$  and the domain of  $g$  is  $\{x|x \neq 0\}$ . Then  $(f \circ g)(x) = \sqrt{\frac{1}{x} + 1}$ , so the domain of  $f \circ g$  is  $\{x|x \leq -1 \text{ or } x > 0\}$ . On the other hand,  $(g \circ f)(x) = \frac{1}{\sqrt{x+1}}$ , so the domain of  $g \circ f$  is  $\{x|x > -1\}$ .

## STYLISTIC POINTS

Sometimes a write-up that is correct and detailed at each step is nonetheless difficult to follow, simply for stylistic reasons. It is worthwhile to make it pleasant for your audience to read your work (especially if your audience happens to be assigning a grade!!) Here are some general points to keep in mind when writing.

- Make your work legible. In particular, help the reader by spacing your work appropriately on the page and by balancing symbols with words.

– **Example.** Let  $f(x) = \frac{x}{(x^2+1)^2}$ . Then  $f(x) = x(x^2 + 1)^{-2}$ , so  $\frac{d}{dx}(f(x)) = 1(x^2 + 1)^{-2} + x\frac{d}{dx}[(x^2 + 1)^{-2}] = (x^2 + 1)^{-2} + x[(-2)(x^2 + 1)^{-3}(2x)] = \frac{1}{(x^2+1)^2} + \frac{-4x^2}{(x^2+1)^3} = \frac{(1-3x^2)}{(x^2+1)^3}$ .

While this is a correct argument, it is frustrating to read. It is preferable to allow some space and to use words in place of some symbols. For example:

Let  $f(x) = \frac{x}{(x^2 + 1)^2}$ . By rewriting  $f$  as a product, we see that

$$f(x) = x(x^2 + 1)^{-2}.$$

We may then use the product rule to differentiate  $f$ , so

$$\begin{aligned}\frac{d}{dx}(f(x)) &= 1(x^2 + 1)^{-2} + x\frac{d}{dx}[(x^2 + 1)^{-2}] \\ &= (x^2 + 1)^{-2} + x[(-2)(x^2 + 1)^{-3}(2x)] \\ &= \frac{1}{(x^2+1)^2} + \frac{-4x^2}{(x^2+1)^3}.\end{aligned}$$

By choosing a common denominator for the terms in the last step above, we see that

$$\frac{d}{dx}(f(x)) = \frac{1 - 3x^2}{(x^2 + 1)^3}$$

- Include transitional phrases to help guide the reader.
  - **Example.** Let  $x \in \mathbf{R}$  and  $y \in \mathbf{R}$ .  $x^2 \geq 0$  and  $y^2 \geq 0$ .  $x^2 + y^2 \geq 0$ .  $x^2 + y^2 = -1$  has no real solution.

While each statement follows from the previous, it would be far easier for the reader to follow the argument if the author were to give some guidance. For example:

Let  $x \in \mathbf{R}$  and  $y \in \mathbf{R}$ . As  $x$  is a real number,  $x^2 \geq 0$ . As  $y$  is a real number,  $y^2 \geq 0$ . Since  $x^2$  and  $y^2$  are each non-negative, their sum is non-negative as well. That is,  $x^2 + y^2 \geq 0$ . Hence there are no real numbers  $x$  and  $y$  for which  $x^2 + y^2 = -1$ .

## ADDITIONAL COMMENTS

- The purpose of written homework is to develop the skills which are essential for understanding and communicating mathematics. Your homework paper is not a certificate proving that you have done the assignment. Rather, it is an exercise in both learning and in reporting what you have learned. An important way of deepening your understanding of mathematics is to explain it to another person.
- Your primary responsibility is to communicate with the reader. Do not write to the instructor; she already knows how to do the problem. Instead, assume the reader is someone who needs your help. A good guideline is to include details to the extent that, should you find your work again in five years, you could understand it fully on the first reading.
- Make sure your write-up is clear, logical and complete. Your arguments should lead the reader from one step to the next without any leaps of faith or verbal contortions.
- Good writing is not easy; it takes practice. Even professional writers constantly revise and rewrite their work. So, *reread* and *rewrite* your paper. It is rare for a first (or even second) draft to achieve the goals laid out above. Never plan on handing in a first draft.

## ACKNOWLEDGEMENTS

This handout was prepared by Eliza Berry and Jeff Lawson, Fall, 1996. It is a revision of a document prepared by Eliza Berry and Jo-Ann Cohen, Spring, 1996, which in turn was based on a similar document written by Professor J.J. Price of Purdue University.