Exploring Uses of Persistent Homology for Hyperspectral Remote Sensing

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A Better Title

An Application of Persistent Homology on Grassmann Manifolds to the Detection of Signals in Hyperspectral Imagery

Introduction

- **Topological data analysis (TDA)**

Data points → Geometric object → Topological summary.

- **Persistent Homology (PH)** can be applied to a data set to capture the "persistence" of topological structure.

- We apply PH to **hyperspectral data** encoded as abstract points on a **Grassmann manifold** $G(k,n)$.

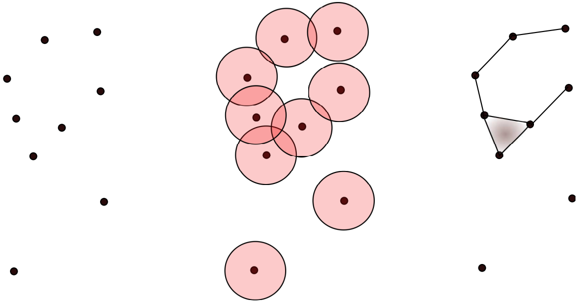
- $G(k,n)$ **framework** affords a form of compression while retaining topological structure → it becomes feasible to analyze **large volumes of hyperspectral data**.

Persistent Homology

- Understand the overall shape of a given point cloud data
- Homology: topological space → vector space
- Persistent homology: nested sequence of subspaces → sequence of vector spaces
- Intuitively: can identify clusters, holes, voids, etc. in a point cloud.

Persistent Homology

Finite data set → Simplicial complex (Vietoris-Rips)

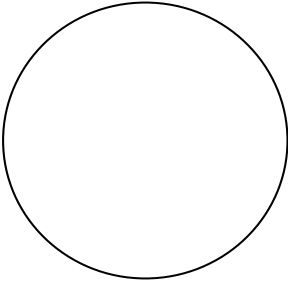


Persistent Homology

i-th Betti number = rank of the i-th homology of a simplicial complex:

- $Betti_0$: 0th order holes, or clusters (connected components).
- $Betti_1$: 1st order holes, or holes (circles).
- $Betti_2$: 2nd order holes, or cavities (voids).

Our Intuition: Circle



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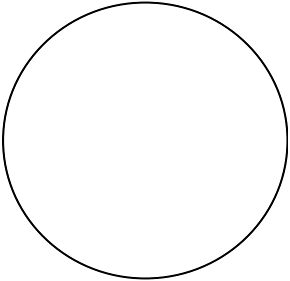
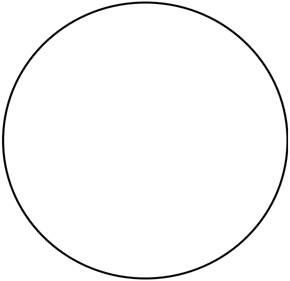


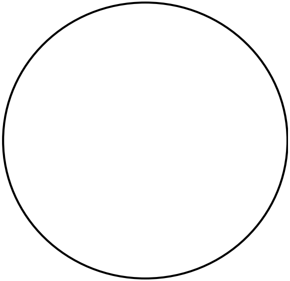
- $Betti_0 = 1$

Our Intuition: Circle



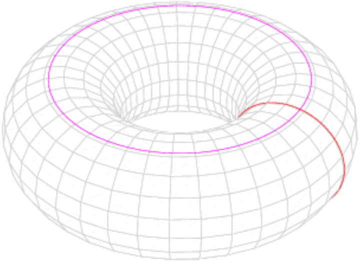
- $Betti_0 = 1$
- $Betti_1 = 1$

Our Intuition: Circle

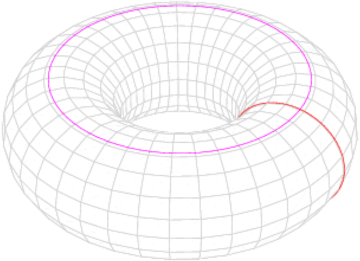


- $Betti_0 = 1$
- $Betti_1 = 1$
- $Betti_2 = 0$

Our Intuition: Torus

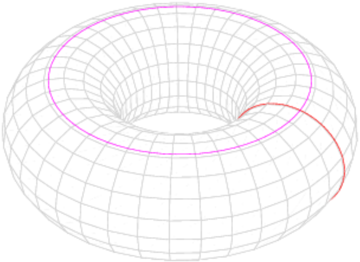


Our Intuition: Torus



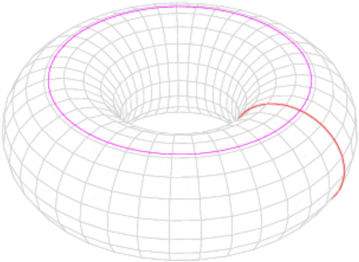
- $Betti_0 = 1$

Our Intuition: Torus



- $Betti_0 = 1$
- $Betti_1 = 2$

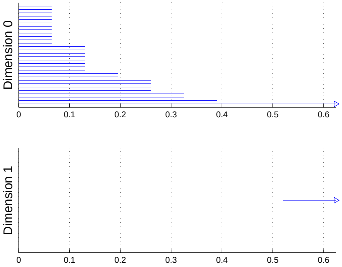
Our Intuition: Torus



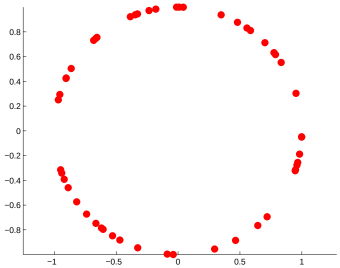
- $Betti_0 = 1$
- $Betti_1 = 2$
- $Betti_2 = 1$

Barcodes

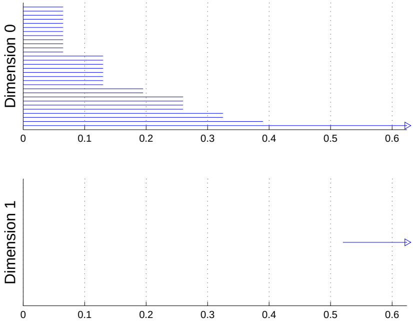
- Persistent homology tracks homology classes along the filtration: at what value of parameter does a hole appear, and how long does it persist?
- Each horizontal bar represents the birth-death of a separate homology class.
- The i-th Betti number at any given parameter value is the number of bars.



Barcodes

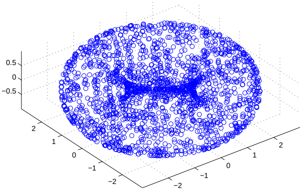


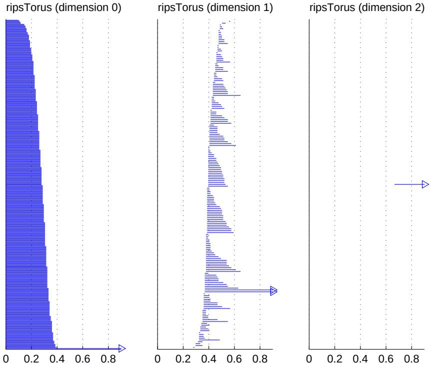
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−0.5
0
0.5
1



Dimension 0
0
0.1
0.2
0.3
0.4
0.5
0.6
Dimension 1
0
0.1
0.2
0.3
0.4
0.5
0.6

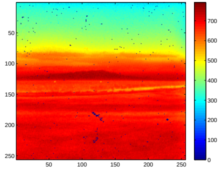
Barcodes

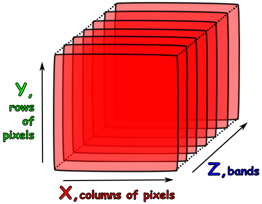

0.5
0
−0.5
2
1
0
−1
−2
2
1
0
−1
−2


ripsTorus (dimension 0)
ripsTorus (dimension 1)
ripsTorus (dimension 2)
0 0.2 0.4 0.6 0.8
0 0.2 0.4 0.6 0.8
0 0.2 0.4 0.6 0.8

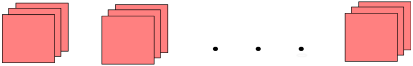
Fabry-Pérot Long-Wavelength Infrared Data Set

- **Single scanning:** 20 images from different wavelengths ($8 - 11\ \mu$m), each image is 256×256 pixels.





- **Triethyl Phosphate (TEP):** 561 data cubes $256 \times 256 \times 20$.
- **Data collection:** The cubes are collected successively to record the event from 'pre-burst' to 'post-burst'.



Grassmannian Framework

Grassmann manifold

The real $G(k,n)$ is the collection of all k-dimensional subspaces of $\mathbb{R}^n$, $k \leq n$.
$G(k,n) = O(n)/(O(k) \times O(n-k))$.

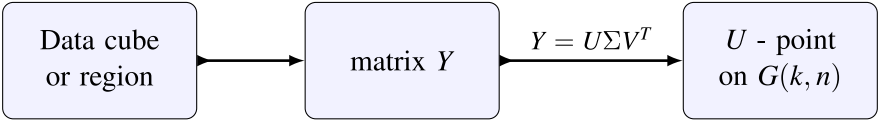
Representation

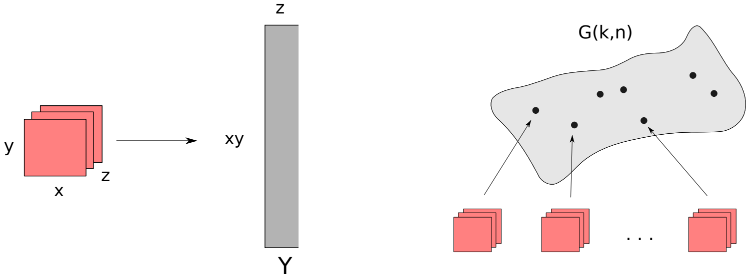
A point on $G(k,n)$ can be represented by an $n \times k$ orthogonal matrix U ($U^T U = I_k$).

Example

$G(1,n)$ is the set of all lines going through the origin of $\mathbb{R}^n$ (*projective space* $\mathbb{RP}^{n-1}$).

Grassmannian Framework

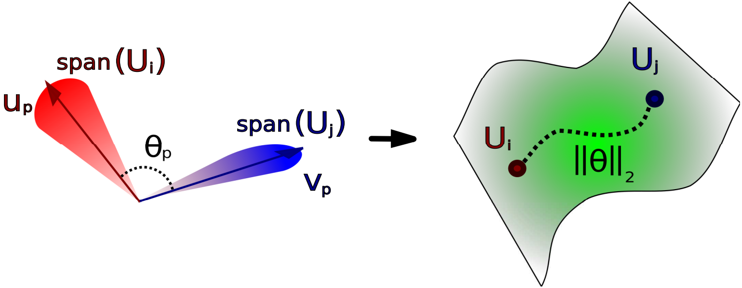




- $256 \times 256 \times 20 \rightarrow$ a point on $G(20, 65536)$.
- $4 \times 8 \times 20 \rightarrow$ a point on $G(20, 32)$.

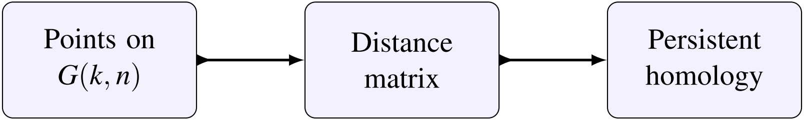
Distance Metrics on $G(k,n)$:

- Geodesic or arc length: $d_G(U_i, U_j) = \|\theta\|_2$
- Pseudometric: $d_G(U_i, U_j) = \theta_1$



Principal angles: $0 < \theta_1 \leq \theta_2 \leq \ldots \leq \theta_k \leq \pi/2$,
$\cos\theta_p = \underset{u_p \in span(U_i)}{max} \underset{v_p \in span(U_j)}{max} u_p^T v_p$,
s. t. $u_p^T u_p = 1, v_p^T v_p = 1,\ u_p^T u_q = 0, v_p^T v_q = 0,\ q = 1, \ldots, p-1.$

Summary



Experiment

- We use 3 (out of 20) bands selected by sparse support vector machine algorithm.
- A patch $4 \times 8 \times 3$ from each cube is mapped to a point on $G(3, 32)$.
- The location of the patch is determined by ACE detector.
- We generate $Betti_0$ barcodes to see clusters in data.

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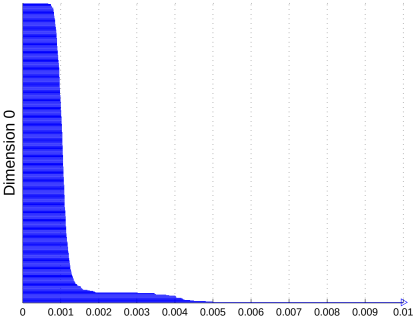
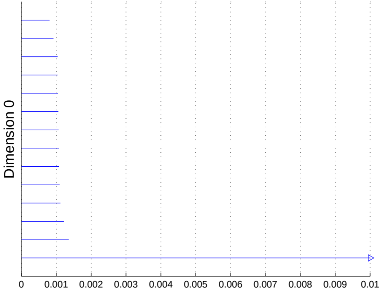
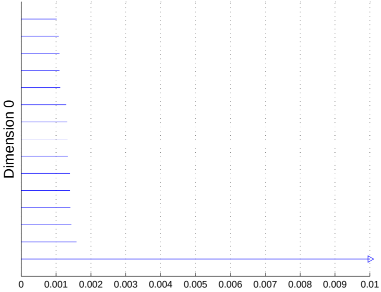


Experiment

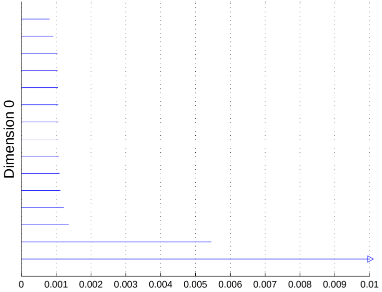
Background only points:

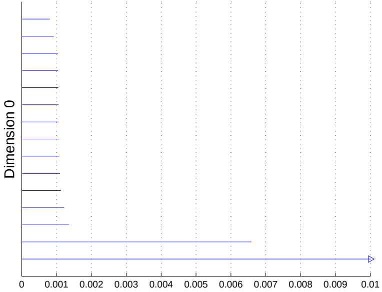




Experiment

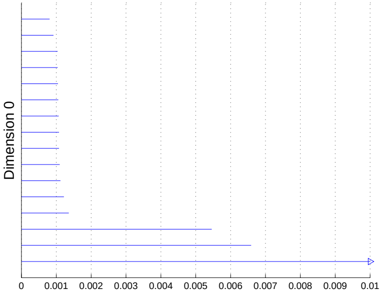
Points before TEP release and one TEP point:

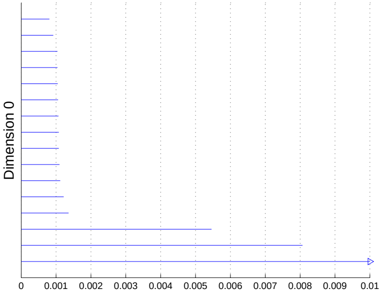




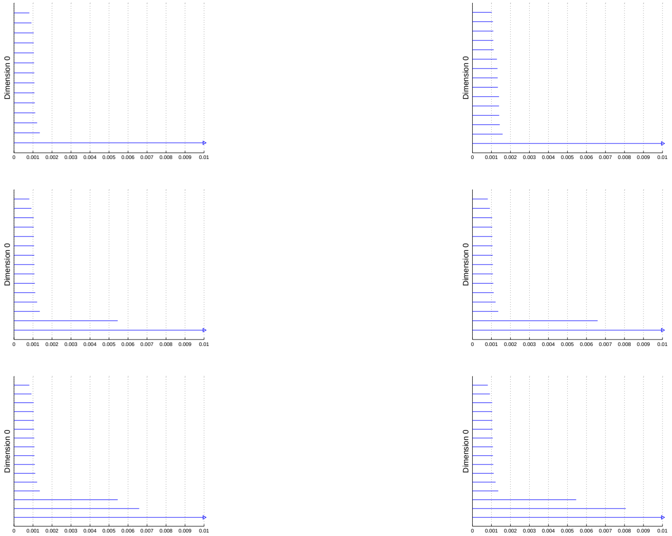
Experiment

Points before TEP release and two TEP points:





Experiment


Dimension 0
0 0.001 0.002 0.003 0.004 0.005 0.006 0.007 0.008 0.009 0.01

Future Work: Strengthening the Topological Signal

- Employ different data settings.
- Use other distance (pseudo) metrics on the Grassmannian.
- Explore $Betti_1$ barcodes for analysis.

References

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