Formula Sheet

- Table of Laplace Transforms

Functional Properties:

$f(t) = \mathfrak{L}^{-1}(F(s))$	$F(s) = \mathfrak{L}(f(t)) = \int_0^\infty e^{-st} f(t)\ dt$
$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$
$tf(t)$	$-\frac{d}{ds}\Big[F(s)\Big]$
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}\Big[F(s)\Big]$
$e^{at} f(t)$	$F(s-a)$
$u(t-c)f(t-c)$ (for $c \geq 0$)	$e^{-cs} F(s)$
$u(t-c)f(t)$ (for $c \geq 0$)	$e^{-cs} \mathfrak{L}(f(t+c))$

Specific Transforms:

$f(t) = \mathfrak{L}^{-1}(F(s))$	$F(s) = \mathfrak{L}(f(t))$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
te^{at}	$\frac{1}{(s-a)^2}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$u(t-c)$ (for $c \geq 0$)	$\frac{e^{-cs}}{s}$

- For a differential equation of the form

$$\frac{dx}{dt} + p(t)x(t) = q(t),$$

  the general solution is in the form

$$x(t) = \frac{1}{\mu(t)}\Big(\int \mu(t)q(t)\ dt + C\Big),$$

  where

$$\mu(t) = e^{\int p(t)dt}.$$

- For a differential equation of the form

$$\frac{dx}{dt} = p(t)x + q(t)x^n,\ \ n \neq 0, 1,$$

  use the substitution $v(t) = (x(t))^{1-n}$.

- Newton's Law of Cooling: $T'(t) = k(A - T(t))$.

- Logistic growth equation: $P'(t) = rP(t)(1 - P(t)/N)$.

- A differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

  is exact if and only if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. The general solution of an exact equation is of the form $F(x, y) = C$ where $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$.

- For an initial value problem $x'(t) = f(t, x)$, $x(t_0) = x_0$, Euler's Method finds approximation of solution by

$$x_{i+1} = x_i + f(t_i, x_i)\Delta t$$

$$t_{i+1} = t_i + \Delta t$$

  $(i = 1, 2, 3, \dots)$, given a stepsize Δt.

- Errors in numerical methods:

Method	*Error*
Euler's	$\sim \Delta t$
Improved Euler's	$\sim (\Delta t)^2$
4th-Order Runge-Kutta	$\sim (\Delta t)^4$

- The Wronskian of two solutions $x_1(t)$ and $x_2(t)$ of the second-order linear homogeneous equation $x''+p(t)x'+q(t)x=0$ is the determinant

$$W(x_1,x_2)(t)=\begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix}.$$

  If $W(x_1,x_2)(t)\neq 0$ for all values of t, then $x_h(t)=C_1x_1(t)+C_2x_2(t)$ is a general solution of $x''+p(t)x'+q(t)x=0$.

- The method of variation of parameters finds a particular solution x_p of the nonhomogeneous equations $x''+p(t)x'+q(t)x=f(t)$ in the form $x_p(t)=v_1(t)x_1(t)+v_2(t)x_2(t)$ where $\{x_1(t),x_2(t)\}$ is a fundamental solution set to the homogeneous equation $x''+p(t)x'+q(t)x=0$ and

$$v_1(t)=-\int \frac{x_2(t)f(t)}{W(x_1,x_2)(t)}\,dt$$

  and

$$v_2(t)=\int \frac{x_1(t)f(t)}{W(x_1,x_2)(t)}\,dt$$

- For the Cauchy-Euler equation $at^2x''+btx'+cx=0$, using the substitution $s=\ln t$ (this means that $t=e^s$) with $s'(t)=1/t=e^{-s}$ and letting $x(t)\equiv Y(s)$, the original equation becomes the linear equation $aY''(s)+(b-a)Y'(s)+cY(s)=0$.