Formula Sheet

- Trigonometric identities:

  $\sin^2 x + \cos^2 x = 1$ $\qquad$ $\cos 2x = \cos^2 x - \sin^2 x$
  $\tan^2 x + 1 = \sec^2 x$ $\qquad$ $\sin^2 x = (1 - \cos 2x)/2$
  $\sin 2x = 2 \sin x \cos x$ $\qquad$ $\cos^2 x = (1 + \cos 2x)/2$

- The line through the point (x_0, y_0, z_0) parallel to the vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is given by

$$\mathbf{r}(t) = (x_0 + tv_1)\mathbf{i} + (y_0 + tv_2)\mathbf{j} + (z_0 + tv_3)\mathbf{k}$$

- The plane through the point (x_0, y_0, z_0) with normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is given by the equation

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

- $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$ and $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$
- The arc length of a curve $\mathbf{r}(t)$ for $a \le t \le b$ is given by

$$L = \int_a^b |\mathbf{v}(t)| dt,$$

  where $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$.

- Distance from point S to a line passing through point P, parallel to a vector $\mathbf{v}$: $d = \frac{|\mathbf{PS} \times \mathbf{v}|}{|\mathbf{v}|}$.
- Given point S in space, and a plane with normal $\mathbf{n}$ and point P on the plane, the distance from S to the plane is $d = \frac{|\mathbf{PS} \cdot \mathbf{n}|}{|\mathbf{n}|}$.
- Principal unit normal vector $\mathbf{N} = (d\mathbf{T}/dt)/|d\mathbf{T}/dt|$ (where $\mathbf{T}$ is the unit tangent vector).
- Curve curvature $\kappa = (1/|\mathbf{v}|)|d\mathbf{T}/dt|$.
- Acceleration $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$, where $a_T = \frac{d(|\mathbf{v}|)}{dt}$ and $a_N = \sqrt{|\mathbf{a}|^2 - a_T^2}$.
- For a plane curve given by $x = f(t)$ and $y = g(t)$, the derivative is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

  provided $dx/dt \neq 0$.

- Areas of surfaces of revolution:

  about x-axis ($y \ge 0$): $S = \int_a^b 2\pi y \sqrt{[dx/dt]^2 + [dy/dt]^2}\ dt$
  about y-axis ($x \ge 0$): $S = \int_a^b 2\pi x \sqrt{[dx/dt]^2 + [dy/dt]^2}\ dt$

- The area enclosed by a polar curve $r = f(\theta)$ between two angles θ_1 and θ_2 is given by

$$\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta = \frac{1}{2}\int_{\theta_1}^{\theta_2} (f(\theta))^2\, d\theta.$$

- The derivative of the function f in the direction of a unit vector $\mathbf{u}$ at the point P_0 is $D_{\mathbf{u}}f(P_0) = \nabla f(P_0) \cdot \mathbf{u}$.

- The linearization of $f(x, y)$ at a point $P_0 = (x_0, y_0)$ is
$L(x, y) = f(P_0) + f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0)$.

- *The Second Derivative Test for Functions of Two Variables:*
Let $f(x, y)$ be a twice differentiable function, and assume its second partial derivatives are continuous. Let (a, b) be a critical point for f, and define the Hessian of f at (a, b) to be
$H(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$.
  - If $H(a, b) < 0$, then (a, b) is a saddle point.
  - If $H(a, b) > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a local min.
  - If $H(a, b) > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a local max.
  - If $H(a, b) = 0$, the test is inconclusive.

- *Polar Coordinates:*
  - $x = r\cos\theta, \quad y = r\sin\theta, \quad r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}$
  - $dA = r\ drd\theta$

- *Cylindrical Coordinates:*
  - $x = r\cos\theta, \quad y = r\sin\theta, \quad z = z, \quad r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}$
  - $dV = dz\ r\ drd\theta$

- *Spherical Coordinates:*
  - $x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$
  - $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}, \quad r = \rho\sin\phi$
  - $dV = \rho^2 \sin\phi\ d\rho d\phi d\theta$

- The center of mass $(\bar{x}, \bar{y}, \bar{z})$ of a three-dimensional object D with density $\delta(x, y, z)$: $\bar{x} = \frac{\int\int\int_D x\delta dV}{\int\int\int_D \delta dV}, \bar{y} = \frac{\int\int\int_D y\delta dV}{\int\int\int_D \delta dV}, \bar{z} = \frac{\int\int\int_D z\delta dV}{\int\int\int_D \delta dV}$.

- Given a vector field $\mathbf{F} = \langle M, N, P\rangle$ (if in 2D, $\mathbf{F} = \langle M, N\rangle$) and a smooth curve C, parameterized by $\mathbf{r}(t)$:

  Flow/Work/Circulation along/around C is given by:
  in 2D: $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M dx + N dy$;
  in 3D: $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M dx + N dy + P dz$.

  Flux of $\mathbf{F}$ <u>across curve C in the plane</u> is $\oint_C \mathbf{F}\cdot\mathbf{n}\ ds = \oint_C M\ dy - N\ dx$.

- *Component Test for Conservative Vector Fields:*
A vector field $\mathbf{F} = \langle M, N, P\rangle$ is conservative if and only if
$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$.
(also: there is a potential function f so that $\nabla f = \mathbf{F}$ and line integrals are independent of path: $\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$.)

- *Green's Theorem* (R is a region enclosed by a curve C):

  $\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C M dx + N dy = \iint_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\ dxdy = \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{k}\ dxdy$
  $\oint_C \mathbf{F}\cdot\mathbf{n}\ ds = \oint_C M dy - N dx = \iint_R (\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y})\ dxdy = \iint_R \nabla\cdot\mathbf{F}\ dxdy$

- *Surface integral* of $f(x,y,z)$ over surface S, parameterized by $\mathbf{r}(u,v)$ over the region R in the uv-plane (for surface area: $f(x,y,z)=1$):

$$\iint_S f(x,y,z)\, d\sigma = \iint_R f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\, dudv.$$

  Flux of a vector field $\mathbf{F}$ through surface S is

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma = \iint_S \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\, dudv$$

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- *Stokes' Theorem:*
  Let S be a piecewise smooth, oriented surface bounded by a simple, closed, piecewise smooth curve C (oriented counterclockwise w.r.t. the normal direction of S), then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, d\sigma = \iint_S (\nabla\times\mathbf{F})\cdot(\mathbf{r}_u \times \mathbf{r}_v)\, dudv.$$

- *The Divergence Theorem:*
  Let D be a simple solid region in space that is bounded by a closed surface S given the outward orientation, then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma = \iiint_D \nabla\cdot\mathbf{F}\, dV.$$