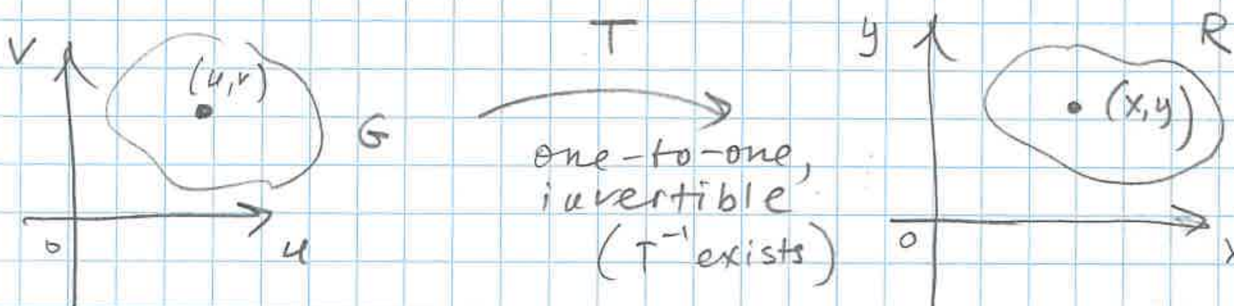
§ 14.8 Substitutions in Multiple Integrals.

In Calculus I, we used u-substitution to replace a complicated integral with an integral that is easier to evaluate.

We can do the same for double integrals

(also, for triple integrals)

We will use a transformation that gives a change of variables from uv-coordinates to xy-coordinates.



region G in uv-plane, preimage of R

region R in xy-plane, image of G under T

$T(u,v) = (x,y)$ by:

$$x = g(u,v)$$
$$y = h(u,v)$$

A function $f(x,y)$ defined on R can now be thought of as a function

$f(g(u,v), h(u,v))$ defined on G.

Q: How we can rewrite $\iint_R f(x,y)\, dA_R$ using the transformation T as the integral of f over preimage G?

$$\boxed{\iint_R f(x,y)\,dA_R = \iint_G f(g(u,v),h(u,v))\underbrace{|J(u,v)|}_{\substack{\text{absolute value of}\\ \text{the Jacobian } J(u,v)}}\,dA_G}$$

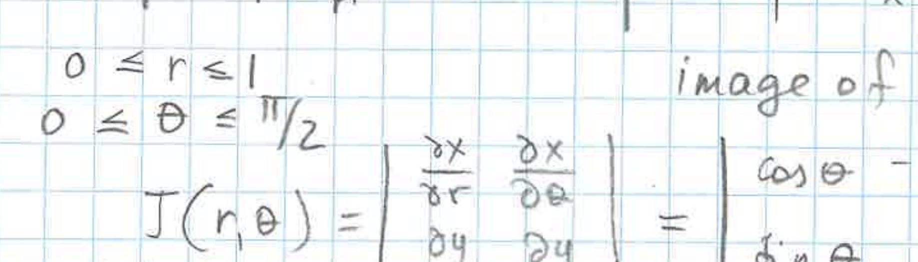
$J(u,v)$ measures the expansion or contraction of the area around a pt. (u,v) in G as G is mapped to R under T.

Definition: The Jacobian (determinant) of the coordinate transformation $x=g(u,v)$, $y=h(u,v)$ is

$$J(u,v) = \underbrace{\frac{\partial(x,y)}{\partial(u,v)}}_{\text{also notation}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

$$\left(\overset{\text{or}}{=} \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - y_u x_v \right)$$

Example 1 Converting to Polar Form.



1st quadrant $x^2+y^2 \le 1$

$T(r,\theta) = (x,y)$:

$$\underbrace{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}_{\text{transform } G \text{ into } R!}$$

$0 \le r \le 1$

$0 \le \theta \le \pi/2$

image of G

$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$\left(\text{or } \frac{\partial(x,y)}{\partial(r,\theta)}\right)$

So, $\iint_R f(x,y)\, dA_R = \iint_G f(r\cos\theta, r\sin\theta)\, |r| \underbrace{dr\,d\theta}_{dA_G}$

Assuming $r \geq 0 \Rightarrow |r| = r$

So, we have $\iint_G f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta \rightarrow$

the polar form of a double integral seen before!

Example 2: Given $u = x + 2y$, $v = x - y$, find $J(u,v)$: first, find $x = x(u,v)$ & $y = y(u,v)$.

$u = x + 2y \Rightarrow 2y = u - x$
$v = x - y \Rightarrow x = v + y$ $\Rightarrow 2y = u - (v+y)$

$= u - v - y \Rightarrow 3y = u - v \Rightarrow y = \frac{u-v}{3}$. Then

$x = v + y = v + \frac{u-v}{3} = \frac{2}{3}v + \frac{1}{3}u$.

The Jacobian $J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

$= \begin{vmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{vmatrix} = -1/9 - \frac{2}{9} = \boxed{-\frac{1}{3}}$

Example 3: R is the region in the 1st quadrant of the xy-plane bounded by the hyperbolas $xy = 1$ & $xy = 9$ and the lines $y = x$ & $y = 4x$. Use the transformation $x = u/v$, $y = uv$ $(u > 0, v > 0)$ to evaluate

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$\iint\limits_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\,dy$. Sketch R and its preimage.

Solution: Note that $\sqrt{\frac{y}{x}} + \sqrt{xy} =$

$u>0, v>0$ $\Big\{ = \sqrt{\frac{uv}{u/v}} + \sqrt{\frac{u}{v}uv} = v + u = f(u,v)$

Jacobian $J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/v & -u/v^2 \\ v & u \end{vmatrix}$

(or $\frac{\partial(x,y)}{\partial(u,v)}$)

$= \frac{u}{v} - \left(-\frac{uv}{v^2}\right) = \frac{u}{v} + \frac{u}{v} = \frac{2u}{v}$ (>0 since $u,v>0$)

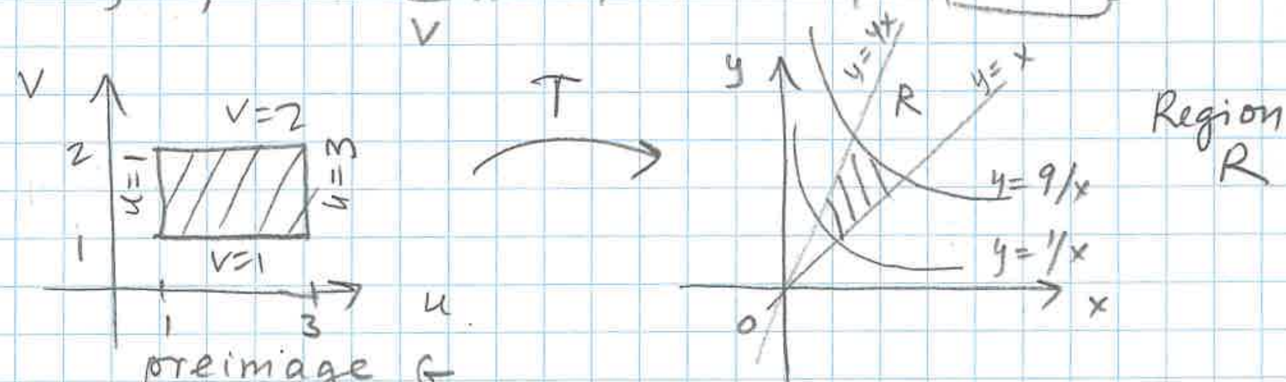
Preimage G:

$y = x \Rightarrow uv = \frac{u}{v} \Rightarrow v^2 = 1 \Rightarrow \boxed{v=1}$

$y = 4x \Rightarrow uv = \frac{4u}{v} \Rightarrow v^2 = 4 \Rightarrow \boxed{v=2}$ $\Big\}\ v>0$

$xy = 1 \Rightarrow \frac{u}{v}uv = 1 \Rightarrow u^2 = 1 \Rightarrow \boxed{u=1}$

$xy = 9 \Rightarrow \frac{u}{v}uv = 9 \Rightarrow u^2 = 9 \Rightarrow \boxed{u=3}$ $\Big\}\ u>0$



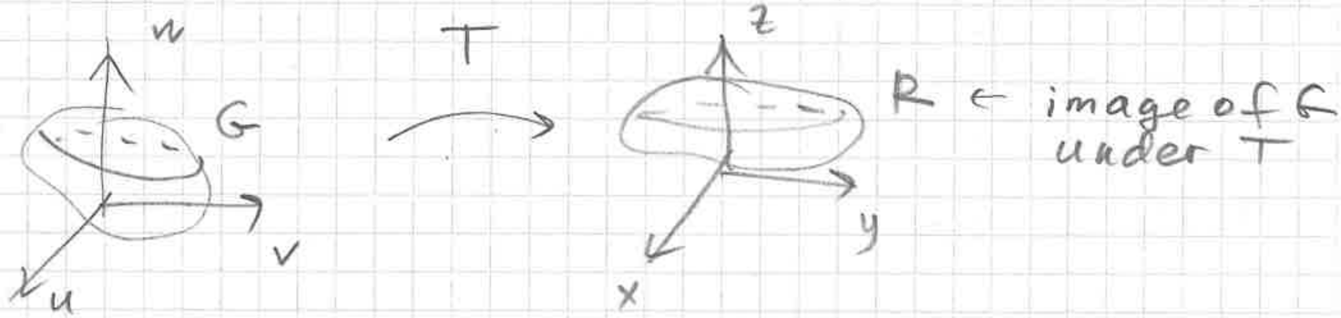
So, $\iint\limits_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\,dy = \int_1^3\int_1^2 (v+u)\overbrace{\frac{2u}{v}}^{>0}\,dv\,du$

$= \int_1^3 \left[2uv + 2u^2 \ln v\right]_1^2 du = \int_1^3 (2u + 2u^2 \ln 2)\,du = \boxed{8 + \frac{52}{3}\ln 2}$

Q: What about substitutions in triple integrals?

Consider G - a region in the uvw-space and let T be transformation from (u,v,w) to (x,y,z) w/ $x = g(u,v,w)$, $y = h(u,v,w)$, and $z = k(u,v,w)$.

$$T: G \to R$$



G - preimage of R under T

The Jacobian is:

$$J(u,v,w) \overset{\text{or}}{=} \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

and the integral under this transfomation is

$$\iiint_R f(x,y,z)\, dV_R = \iiint_G f(g(u,v,w), h(u,v,w), k(u,v,w)) \times |J(u,v,w)|\, dV_G$$

Cylindrical coord's: $(r,\theta,z) \xrightarrow{T} (x,y,z) \Rightarrow$

$$J(r,\theta,z) = r$$

Spherical coord's: $(\rho,\varphi,\theta) \xrightarrow{T} (x,y,z) \Rightarrow$

$$J(\rho,\varphi,\theta) = \rho^2 \sin\varphi.$$