§ 12.3 Arc Length in Space.

(§ 12.3 & § 12.4 are on math features of a curve's shape)

Recall: (Ch. 10) $\mathbb{R}^2$ $x = f(t)$, $y = g(t)$, $a \le t \le b$ $\Rightarrow$ the length of the curve is given by

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt.$$

$\mathbb{R}^3$: consider a smooth curve $\vec{r}(t)$:

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

recall: $\frac{d\vec{r}}{dt}$ is continuous and $\frac{d\vec{r}}{dt} \neq 0$ (no breaks, cusps, corners)

Recall that $\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$

$$|\vec{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Def: The length of a smooth curve $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, $a \le t \le b$, that is traced exactly once as t goes from a to b, is

$$L = \int_a^b |v(t)|\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Example 1 Helix $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + (t)\vec{k}$

Find the length of the helix from the pt. $(1,0,0)$ to pt. $(1,0,2\pi)$

$$L = \int_0^{2\pi} \left|\frac{d\vec{r}}{dt}\right| dt = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{2}\, dt = \boxed{2\sqrt{2}\pi}$$

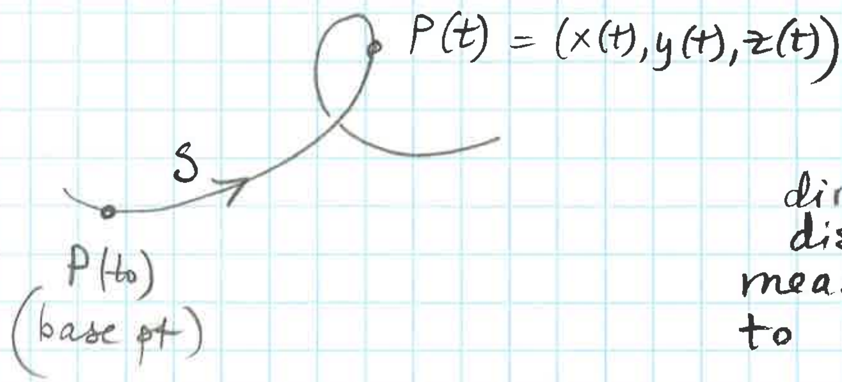
Note: $(1,0,0) \Rightarrow t = 0$
$(1,0,2\pi) \Rightarrow t = 2\pi$

- Length can be found as a function of t for any pt. on curve starting from a base pt. $P(t_0) = (x(t_0), y(t_0), z(t_0))$ by

$$\boxed{s(t) = \int_{t_0}^{t} |\vec{v}(\tau)|\, d\tau}$$

directed distance: measured from $P(t_0)$ to a pt. $P(t)$

dummy variable



$t > t_0 \Rightarrow s(t) =$ distance along the curve from $P(t_0)$ to $P(t)$

$t < t_0 \Rightarrow s(t) = \ominus$ distance

$s(t)$ values determine pts on the curve
$\Rightarrow$ this parameterizes the curve w.r.t. $\underline{s}$.

s is called an <u>arc length parameter</u> for the curve.

$s \nearrow$ as $t \nearrow$

If $\vec{r}(t)$ determines a curve, and $s(t)$ is the arc length function of $t \Rightarrow$ one can solve for t as a func. of s: $t = t(s) \Rightarrow$

$\vec{r} = \vec{r}(t(s))$ gives a new parameterization of the curve in terms of s.

<u>Example 2</u>: Take $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + (t)\vec{k}$ (helix)

& $t_0 = 0$. Then $\underset{\text{arc length from } t_0 \text{ to } t}{s(t)} = \int_{t_0}^{t} \underbrace{|\vec{v}(\tau)|}_{\sqrt{2}}\, d\tau$ $\longrightarrow$ see Example 1

$= \int_0^t \sqrt{2}\, d\tau = \sqrt{2}\, t \Rightarrow t = \dfrac{s}{\sqrt{2}} \Rightarrow$

Substituting into $\vec{r}$ gives:

$$\vec{r}(t(s)) = \left(\cos\frac{s}{\sqrt{2}}\right)\vec{i} + \left(\sin\frac{s}{\sqrt{2}}\right)\vec{j} + \left(\frac{s}{\sqrt{2}}\right)\vec{k}.$$

Note: $\frac{ds}{dt} = |\vec{v}(t)|$ (FTC) $\left[\frac{d}{dt}\int_{t_0}^{t}|\vec{v}(\tau)|d\tau = |\vec{v}(t)|\right]$

Since $\vec{r}$ is a smooth curve, $\frac{ds}{dt}\left(= |\vec{v}(t)|\right) > 0$ (never zero) $\Rightarrow$ again, $s \nearrow$ (as $t \nearrow$)

Unit Tangent Vector: Curve $\vec{r}(t)$

$\vec{v} = \frac{d\vec{r}}{dt}$ is the tangent vector to the curve.

$\vec{v} = \underbrace{|\vec{v}|}_{\text{speed}} \underbrace{\left(\frac{\vec{v}}{|\vec{v}|}\right)}_{\text{direction of motion}}$. We call $\boxed{\vec{T} = \frac{\vec{v}}{|\vec{v}|}}$ the **unit tangent vector**

Example 3: $\vec{r}(t) = (1+3\cos t)\vec{i} + (3\sin t)\vec{j} + t^2\vec{k}$

$\Rightarrow \vec{v}(t) = (-3\sin t)\vec{i} + (3\cos t)\vec{j} + (2t)\vec{k}$

$|\vec{v}(t)| = \sqrt{9\sin^2 t + 9\cos^2 t + 4t^2} = \sqrt{9+4t^2} \Rightarrow$

$$\vec{T} = \left(\frac{-3\sin t}{\sqrt{9+4t^2}}\right)\vec{i} + \left(\frac{3\cos t}{\sqrt{9+4t^2}}\right)\vec{j} + \left(\frac{2t}{\sqrt{9+4t^2}}\right)\vec{k}$$

- $\boxed{\text{Q:}}$ How does the position vector $\vec{r}(t)$ change w.r.t. arc length s? That is, what is $\frac{d\vec{r}}{ds}$?

Recall: $\frac{ds}{dt} = |\vec{v}(t)| > 0 \Rightarrow s\nearrow \Rightarrow s$ is one-to-one & invertible

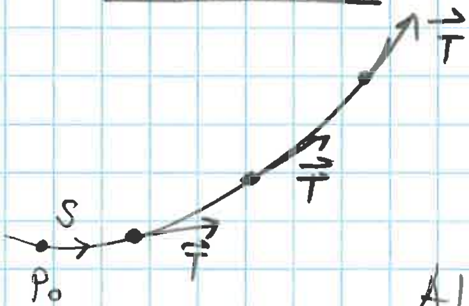
$\Rightarrow t$ is a differentiable func. of s w/ $\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}}$

$= \frac{1}{|\vec{v}(t)|} \Rightarrow \frac{d\vec{r}}{ds} = \underbrace{\frac{d\vec{r}}{dt}\cdot\frac{dt}{ds}}_{\text{chain rule}} = \vec{v}\cdot\frac{1}{|\vec{v}|} = \vec{T}$, i.e., $\boxed{\frac{d\vec{r}}{ds} = \vec{T}}$

§12.4 Curvature & Normal Vectors of a Curve

Curvature

$\mathbb{R}^2$ or $\mathbb{R}^3$



Smooth curve $\vec{r}(t)$ $\left(\frac{d\vec{r}}{dt} \text{ cont.}, \frac{d\vec{r}}{dt} \neq \vec{0} \right)$

$\vec{T} = \dfrac{\vec{v}}{|\vec{v}|}$ unit tangent vector

Also: $\vec{T} = \dfrac{d\vec{r}}{ds}$, where $\underbrace{s(t) = \int_{t_0}^{t} |\vec{v}(\tau)|\, d\tau}_{\text{arc length}}$

$\vec{T}$ turns as the curve bends.

$|\vec{T}| = 1$ (constant), only the direction changes.

Curvature is the rate at which $\vec{T}$ turns per unit of length along the curve, i.e.

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| \qquad (\kappa \text{ - "kappa"})$$

κ large: the curve turns sharply
κ small: the curve turns slowly

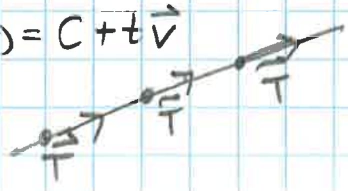
$$\kappa = \left| \frac{d\vec{T}}{ds} \right| = \underbrace{\left| \frac{d\vec{T}}{dt} \frac{dt}{ds} \right|}_{\text{chain rule}} = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right| \qquad \left(\text{since } \frac{ds}{dt} = |\vec{v}| \text{ and } \frac{dt}{ds} = 1/(ds/dt) \right)$$

Formula for κ: $\boxed{\kappa = \dfrac{1}{|\vec{v}|} \left| \dfrac{d\vec{T}}{dt} \right|}$ (a number)

$\left(\text{where } \vec{T} = \dfrac{\vec{v}}{|\vec{v}|} \right)$

Example 1 Curvature of a straight line?

$\vec{r}(t) = C + t\vec{v}$



$\vec{T}$ is a const. vector $\Rightarrow \dfrac{d\vec{T}}{ds} = \vec{0}$

$\Rightarrow \kappa = |\vec{0}| = 0.$

Example 2. Curvature of a circle of radius a? ($\mathbb{R}^2$)

$$\vec{r}(t) = (a\cos t)\vec{i} + (a\sin t)\vec{j} \Rightarrow \vec{v}(t) = (-a\sin t)\vec{i} + (a\cos t)\vec{j}$$

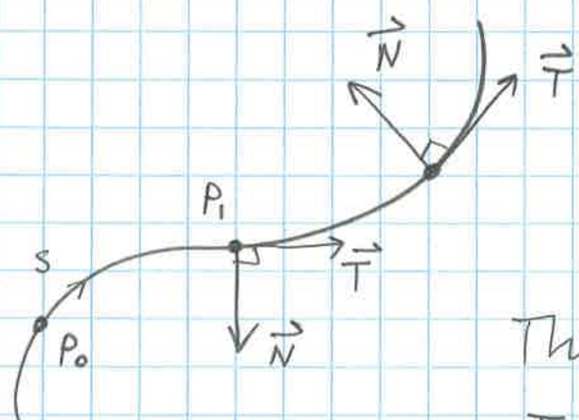
$$\Rightarrow |\vec{v}| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = |a| = a \qquad (a>0)$$

$$\Rightarrow T = \frac{\vec{v}}{|\vec{v}|} = (-\sin t)\vec{i} + (\cos t)\vec{j} \Rightarrow \frac{d\vec{T}}{dt} = (-\cos t)\vec{i} - (\sin t)\vec{j}$$

$$\Rightarrow \left|\frac{d\vec{T}}{dt}\right| = \sqrt{(-\cos t)^2 + (-\sin t)^2} = 1 \Rightarrow \kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \boxed{\frac{1}{a}}$$

or $\kappa = \dfrac{1}{\text{radius}}$

Normal vector



Recall from §12.1:

$\left\{ \begin{array}{l} \text{if } \vec{r}(t) \text{ has } |\vec{r}(t)| = C \\ \text{then it can be shown} \\ \text{that } \vec{r}(t)\cdot\dfrac{d\vec{r}}{dt} = 0, \text{ i.e.} \\ \vec{r}(t) \perp \dfrac{d\vec{r}}{dt} \end{array} \right.$

Thus, since $|\vec{T}| = 1 \Rightarrow$

$$T \perp \frac{d\vec{T}}{ds} \qquad \left(\vec{T}\cdot\frac{d\vec{T}}{ds} = 0\right)$$

- $\dfrac{d\vec{T}}{ds}$ points in the direction in which the curve is turning.

Recall: $\left|\dfrac{d\vec{T}}{ds}\right| = \kappa$, curvature, so, the unit vector in the direction of $\dfrac{d\vec{T}}{ds}$ is

$$\boxed{\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds}} \longrightarrow \text{principal unit normal vector for a smooth curve.}$$

$(\kappa \neq 0)$

Let us find a more useful formula for $\vec{N}$:

$$\vec{N} = \frac{1}{\kappa}\cdot\frac{d\vec{T}}{ds} = \frac{d\vec{T}/ds}{\underbrace{|d\vec{T}/ds|}_{=\kappa}} = \underbrace{\frac{d\vec{T}/dt\cdot dt/ds}{|d\vec{T}/dt\cdot dt/ds|}}_{\text{chain rule}}$$

$$= \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$\frac{dt}{ds} > 0 \rightarrow$ cancels!

So, $\boxed{\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}}$ ($\vec{T}$ is the unit tangent vector)

<u>Example 3:</u> Find $\vec{T}, \vec{N}$, and κ for the curve

$$\vec{r}(t) = (6\sin 2t)\vec{i} + (6\cos 2t)\vec{j} + (5t)\vec{k}$$

<u>Sol.:</u> $\underbrace{\vec{T} = \frac{\vec{v}}{|\vec{v}|},\quad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}}_{\text{vectors}}, \quad \underbrace{\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|}_{\text{scalar}}$

$$\vec{v} = (12\cos 2t)\vec{i} - (12\sin 2t)\vec{j} + 5\vec{k}$$

$$|\vec{v}| = \sqrt{144\cos^2 2t + 144\sin^2 2t + 25} = \sqrt{169} = 13$$

$$\vec{T} = \left(\frac{12}{13}\cos 2t\right)\vec{i} - \left(\frac{12}{13}\sin 2t\right)\vec{j} + \frac{5}{13}\vec{k}$$

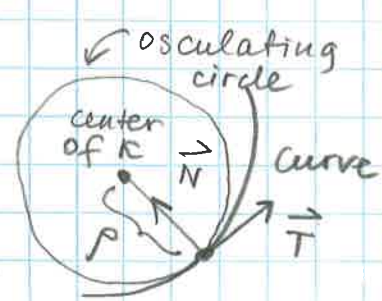
$$\frac{d\vec{T}}{dt} = \left(-\frac{24}{13}\sin 2t\right)\vec{i} - \left(\frac{24}{13}\cos 2t\right)\vec{j} + 0\vec{k}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\left(\frac{24}{13}\right)^2\sin^2 2t + \left(\frac{24}{13}\right)^2\cos^2 2t} = \frac{24}{13}$$

$$\vec{N} = (-\sin 2t)\vec{i} - (\cos 2t)\vec{j}$$

$$\kappa = \frac{1}{13}\cdot\frac{24}{13} = \frac{24}{169}$$

<u>Done!</u>



Note: radius of curvature $\rho = \frac{1}{\kappa}$