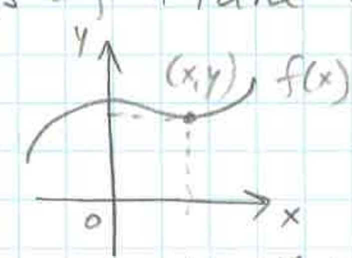
Multivariate (or Multivariable) Calculus.

Ch. 10: Parametric Equations & Polar Coordinates.

Goal: Study new ways to define curves in the plane.

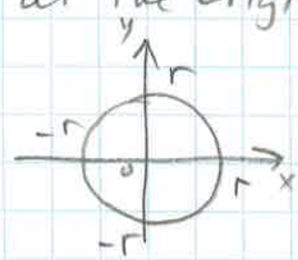
§10.1 Parameterizations of Plane Curves.

Previously: $y = f(x)$



Not all curves fall easily in this form:

for example, a circle centered at the origin of radius r: $x^2+y^2=r^2$



Solve for y or x:

$y = \pm\sqrt{r^2-x^2}$ or $x = \pm\sqrt{r^2-y^2}$

$\downarrow$ 2 functions here!

$y = \sqrt{r^2-x^2}$
$y = -\sqrt{r^2-x^2}$
(top & bottom semicircles)

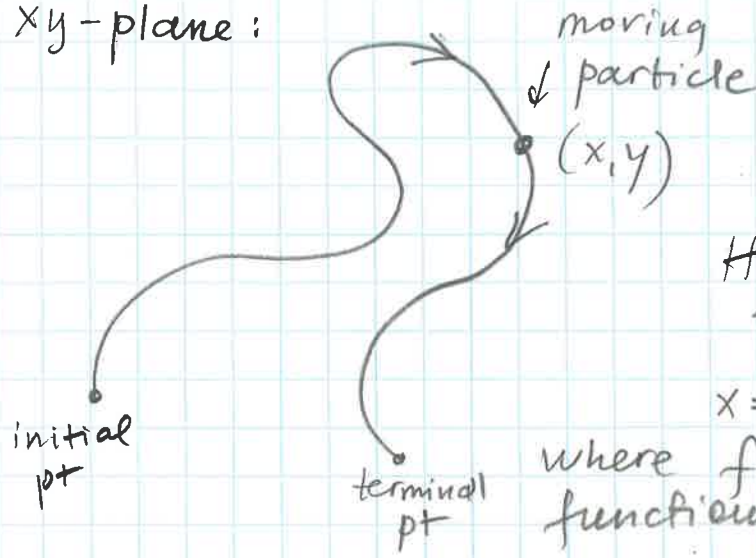
$\downarrow$ 2 functions here!

$x = \sqrt{r^2-y^2}$
$x = -\sqrt{r^2-y^2}$
(right side & left side semicircles)

Such curves can be written down using <u>parametric equations</u>.

A more general example:

xy-plane:



(Note: not a function!)

How do we describe the path? → Using

$x = f(t)$ & $y = g(t)$

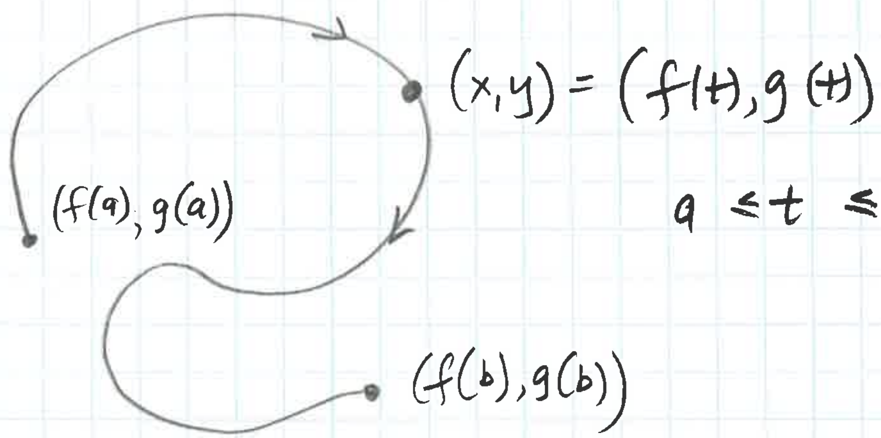
where f & g are continuous functions of t (time).

<u>Def:</u> If x & y are given as functions $x = f(t)$, $y = g(t)$ over an interval I of t-values, then the set of points $(x,y) = (f(t), g(t))$ defined by these equations is a <u>parametric curve.</u>

The equations are <u>parametric equations</u> to the curve.

Each t defines a point $(x,y) = (f(t), g(t))$; t is called a <u>parameter</u> for the curve.

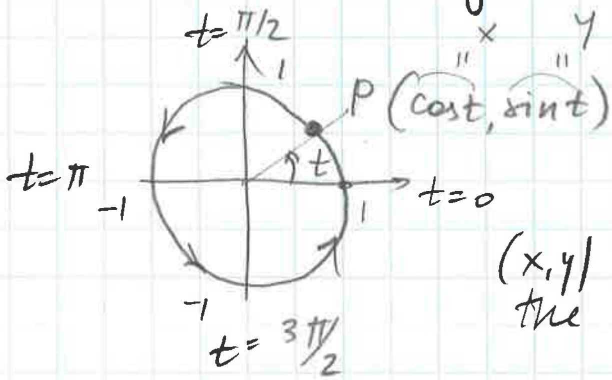
If $I = [a,b]$ (closed), i.e. $a \le t \le b$, then $(f(a), g(a))$ is the <u>initial point</u> and $(f(b), g(b))$ is the <u>terminal point</u> of the curve.



$a \le t \le b$.

Example 1: a) Circle $x^2+y^2=1$ $(r=1)$

Parameterization: $x = \cos t$
$y = \sin t$, $0 \le t \le 2\pi$



$\left(\cos^2 t + \sin^2 t = x^2 + y^2 = 1\right)$

(x,y) starts at $(1,0)$ and traces the curve once counterclockwise as t varies from 0 to 2π

b) circle of radius r: $x^2+y^2=r^2$ $\quad x = r\cos t$, $y = r\sin t$, $0 \le t \le 2\pi$

$\left(x^2+y^2 = r^2\cos^2 t + r^2\sin^2 t = r^2\right)$

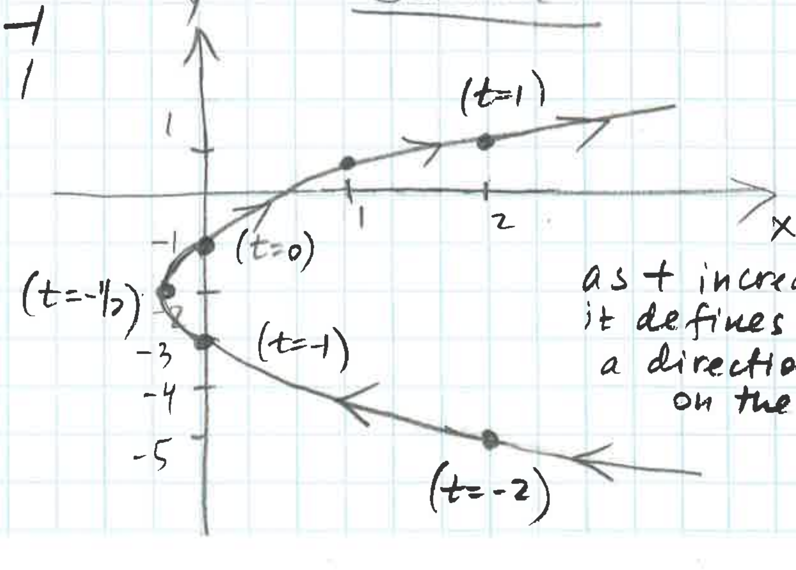
Example 2: Sketch the parametric curve defined by the equations:

$$x = t^2+t, \quad y = 2t-1, \quad -\infty < t < \infty$$

Table of values: (simply pick t's until we have an idea of what the curve looks like)

t	x	y
-2	2	-5
-1	0	-3
$-1/2$	$-1/4$	-2
0	0	-1
1	2	1

Sketch:



(no initial pt & no terminal pt for this curve)

as t increases, it defines a direction on the curve

The curve is a parabola. Indeed, if we eliminate t from equations, we will get:

Solve $y = 2t - 1$ for $t \Rightarrow t = \frac{y+1}{2}$, then plug this into the equation for x:

$$x = t^2 + t = \left(\frac{y+1}{2}\right)^2 + \frac{y+1}{2} = \frac{y^2}{4} + \frac{2y}{4} + \frac{1}{4} + \frac{y}{2} + \frac{1}{2} = \frac{y^2}{4} + y + \frac{3}{4}$$

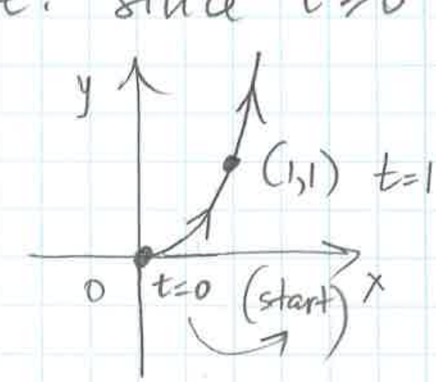
$\Rightarrow$ parabola that opens to the right with a vertex at $(-1/4, -2)$.

__Example 3:__ Identify the curve if it is given by parametric equations

$$x = \sqrt{t},\ y = t,\ t \geq 0$$

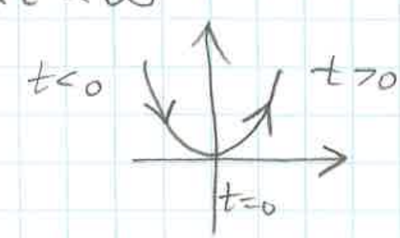
__Solution:__ $y = t = (\sqrt{t})^2 = x^2$ - parabola

Note: since $t \geq 0 \Rightarrow$ it is only half the parabola! $(x, y \geq 0)$



Note that $y = x^2$ can be parameterized by $x = t$, $y = t^2$, $-\infty < t < \infty$

(if $t \geq 0 \Rightarrow$ see above)



Called: natural parameterization $(x = t,\ y = f(t))$

Example 4: Line with slope m, going through the point (a,b):

$y-b=m(x-a)$. If $t=x-a \Rightarrow$

$$x=a+t, \quad y=b+mt, \quad -\infty<t<\infty$$

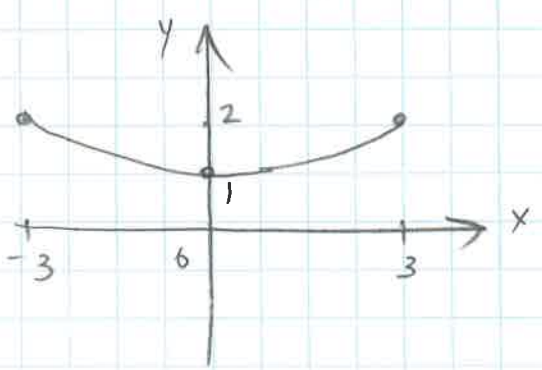
Example 5: What is the path of a particle given by the set of parametric equations:

$$x=3\cos(2t), \quad y=1+\cos^2(2t) \;? \quad (-\infty<t<\infty)$$

$$\cos(2t)=\frac{x}{3} \Rightarrow y=1+\left(\frac{x}{3}\right)^2=1+\frac{x^2}{9} \Rightarrow$$

parabola opening upward. Since

$$-1\le\cos(2t)\le 1 \Rightarrow -3\le 3\cos 2t\le 3 \Rightarrow \boxed{-3\le x\le 3}$$
$$0\le\cos^2(2t)\le 1 \Rightarrow 1\le 1+\cos^2(2t)\le 2 \Rightarrow \boxed{1\le y\le 2}$$



Read (pp. 560-562) about cycloids, brachistochrones & tautochrones.

(See Geogebra links for demo) on the web page.