§10.2 Calculus with Parametric Curves.

Goal: find slopes, lengths, and areas associated with parametric curves.

- A parametric curve $x=f(t)$, $y=g(t)$ is differentiable at t if f & g are differentiable at t.

Q: What is the formula for $\frac{dy}{dx}$ s.t. we could find a slope at any t and equation for tangent line?

Recall: for a function $y=F(x)$, the tangent line at $x=a$ is

$$y = F(a)+m(x-a), \text{ where } m=\left.\frac{dy}{dx}\right|_{x=a}=F'(a)$$

Suppose, we were able to eliminate t from $x=f(t)$ & $y=g(t)$ and got $y=F(x)$.

Then since $x=f(t)$ & $y=g(t)$, $\Rightarrow$

$$g(t) = F(f(t)) \text{ and } g'(t) = \underbrace{F'(f(t))\cdot f'(t)}_{\text{Chain rule!}}$$

Thus, $\underbrace{\frac{dy}{dt}}_{g'(t)} = F'(\underbrace{x}_{=f(t)})\cdot \underbrace{\frac{dx}{dt}}_{f'(t)} \Rightarrow \underbrace{F'(x)}_{\frac{dy}{dx}} = \frac{dy/dt}{dx/dt}$

this is what we needed: since $y=F(x) \Rightarrow$

$$\boxed{\frac{dy}{dx} = F'(x) = \frac{dy/dt}{dx/dt}} \quad \left(\text{provided } \frac{dx}{dt}\neq 0\right)$$

(1)

(Note that we can also have $x = G(y)$
$\Rightarrow \frac{dx}{dy} = \frac{dx/dt}{dy/dt}$, $\frac{dy}{dt} \neq 0$)

Example 1 Find the tangent line to the curve $x = t^5 - 4t^3$, $y = t^2$ at $(0,4)$.

Solution: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{5t^4 - 12t^2} = \frac{2}{5t^3 - 12t}$

What is the t-value when $x=0, y=4$?

$t^5 - 4t^3 = 0 \Rightarrow t^3(t^2-4) = 0 \Rightarrow t = 0, \pm 2$
$t^2 = 4 \Rightarrow t = \pm 2$

So, when $t = \pm 2$, we have $x = 0$, $y = 4$. $\Rightarrow$ i.e., we have two values of t w/ two tangent lines:

$t=-2 \Rightarrow m = \left.\frac{dy}{dx}\right|_{t=-2} = \frac{2}{5(-2)^3 - 12(-2)} = -\frac{1}{8}$

and tangent line is

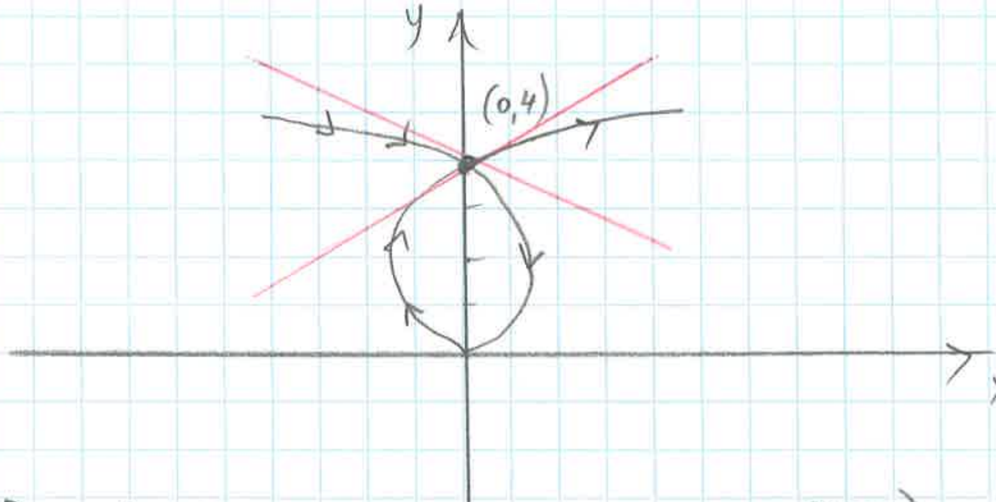
$$y = 4 - \frac{1}{8}(x-0) \Rightarrow \boxed{y = -\frac{1}{8}x + 4}$$

$t=2 \Rightarrow m = \left.\frac{dy}{dx}\right|_{t=2} = \frac{2}{5 \cdot 2^3 - 12 \cdot 2} = \frac{1}{8}$ and tangent line is

$$\boxed{y = \frac{1}{8}x + 4}$$

How is it possible to have 2 tangent lines?

Sketch the curve: it crosses itself!



(Read Example 1 on page 565)

Second Derivative d^2y/dx^2:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \underbrace{\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}}_{\text{formula (1) applied to } \frac{dy}{dx} = y'} \text{ or } = \frac{\frac{d}{dt}(y')}{dx/dt}$$

(Note: $\frac{d^2y}{dx^2} \neq \frac{d^2y}{dt^2} \Big/ \frac{d^2x}{dt^2}$!)

Example 2 Take the same curve

$x = t^5 - 4t^3, \quad y = t^2$

Found earlier: $\frac{dy}{dx} = \frac{2}{5t^3 - 12t}$, Then we first

find $\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{2}{5t^3-12t}\right) =$

$$= \frac{(-2)(15t^2-12)}{(5t^3-12t)^2} = \frac{24-30t^2}{(5t^3-12t)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{24-30t^2}{(5t^3-12t)^2}}{5t^4-12t^2}$$

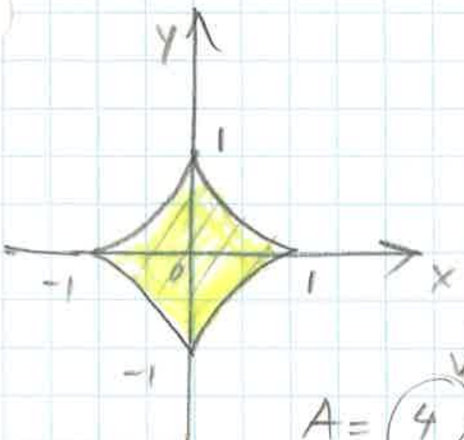
$$= \frac{24-30t^2}{(5t^4-12t^2)(5t^3-12t)^2} = \frac{24-30t^2}{t(5t^3-12t)^3}.$$

(See Example 2 on p. 565)

Area w/ Parametric Curves

Example 3 (p. 565-566)

Find the area enclosed by the astroid $x = \cos^3 t$, $y = \sin^3 t$, $0 \le t \le 2\pi$



Solution: By symmetry, the area is 4 times the area under the curve in the 1st quadrant, where $0 \le t \le \pi/2$ $\Rightarrow$

$$A = 4\int_0^1 y\,dx = 4\int_{\pi/2}^{0} \underbrace{\sin^3 t}_{y(t)}\ \underbrace{3\cos^2 t(-\sin t)dt}_{dx = x'dt}$$

$$= 12\int_0^{\pi/2} \sin^4 t\cos^2 t\,dt$$

($x=0$ when $t=\pi/2$, $x=1$ when $t=0$)

$$= 12\int_0^{\pi/2}\left(\frac{1-\cos 2t}{2}\right)^2\left(\frac{1+\cos 2t}{2}\right)dt$$

$$= \frac{3}{2}\int_0^{\pi/2}(1-2\cos 2t+\cos^2 2t)(1+\cos 2t)\,dt$$

$$= \frac{3}{2}\int_0^{\pi/2}\left(1-\cos 2t - \cos^2 2t + \cos^3 2t\right)dt$$

$$= \frac{3}{2}\left[\int_0^{\pi/2}(1-\cos 2t)\,dt - \int_0^{\pi/2}\cos^2 2t\,dt + \int_0^{\frac{\pi}{2}}\cos^3 2t\,dt\right]$$

$$= \frac{3}{2}\left[\left(t-\frac{1}{2}\sin 2t\right) - \frac{1}{2}\left(t+\frac{1}{4}\sin 2t\right) + \frac{1}{2}\left(\sin 2t - \frac{1}{3}\sin^3 2t\right)\right]_0^{\pi/2}$$

$$= \boxed{3\pi/8}$$

(see Sec. 8.2, p. 430)

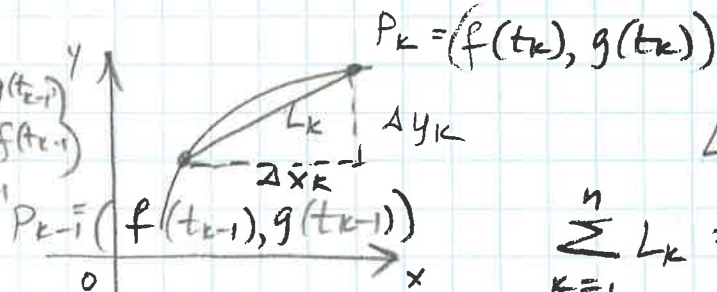
Length of a Parametric Curve

Definition: If a curve C is defined parametrically by $x = f(t)$ & $y = g(t)$, $a \le t \le b$, where f' & g' are continuous and not simultaneously 0 on $[a, b]$, and C is traversed exactly once as t increases from a to b, then the length of C is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt \overset{\text{or}}{=} \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\,dt$$

Why is that? (Recall §6.3)

$\Delta y_k = g(t_k) - g(t_{k-1})$
$\Delta x_k = f(t_k) - f(t_{k-1})$
$\Delta t_k = t_k - t_{k-1}$



$$L_k = \sqrt{\Delta x_k^2 + \Delta y_k^2}$$

$$\sum_{k=1}^n L_k = \sum_{k=1}^n \sqrt{\Delta x_k^2 + \Delta y_k^2}$$

$$\overset{\text{MVT}}{=} \sum_{k=1}^n \sqrt{[f'(t_k^*)]^2 + [g'(t_k^{**})]^2}\,\Delta t_k \Rightarrow \text{take limit}$$

(t_k^*, t_k^{**} from $[t_{k-1}, t_k]$)

(see p. 566-567)

Example 4

Determine the length of the curve defined by $x = 3\sin t$, $y = 3\cos t$, $0 \le t \le 2\pi$

Solution: $\dfrac{dx}{dt} = 3\cos t$, $\dfrac{dy}{dt} = -3\sin t$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{9\cos^2 t + 9\sin^2 t}\, dt$$

$$= 3\int_0^{2\pi} dt = \boxed{6\pi}$$

(Also see examples 4, 5, 6 on pp. 567–569)

Length of a Curve $y = f(x)$

(Section 6.3)

Given $y = f(x)$, $a \le x \le b$, we can assign $x = t$, $a \le t \le b$, $y = f(t)$ $\Rightarrow$ so-called natural parameterization.

Then $\dfrac{dx}{dt} = 1$, $\dfrac{dy}{dt} = f'(t) \Rightarrow \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

$= f'(t) \Rightarrow \left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = 1 + [f'(t)]^2$

$\overset{\shortparallel}{(f'(x))}$ $\quad = 1 + [f'(x)]^2$

or

$\Rightarrow$ So, we get $L = \displaystyle\int_a^b \sqrt{1 + (f'(x))^2}\, dx$

(arc length) $\quad = \displaystyle\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$.

(saw it in Calc. 2)

The Arc Length Differential

$$s(t) = \int_a^t \sqrt{[f'(z)]^2 + [g'(z)]^2}\, dz$$

$$a \le t \le b$$

arc length function $\Rightarrow$ by FTC,

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \Rightarrow ds = \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}_{\frac{ds}{dt}}\, dt$$

$$\Rightarrow ds = \sqrt{dx^2 + dy^2} \Rightarrow$$

$$L = \int_a^b ds$$

(Examples 7, 8 p. 570–571)

Area of Surfaces of Revolution

(Also see Sec. 6.4)

$x = f(t)$, $y = g(t)$, $a \le t \le b$. If the curve is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about

1) the x-axis ($y \ge 0$):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \overset{\text{or}}{=} \int_a^b 2\pi y\, ds$$

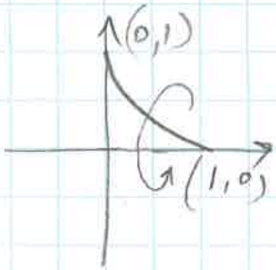
2) the y-axis ($x \ge 0$):

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \overset{\text{or}}{=} \int_a^b 2\pi x\, ds$$

<u>Example 5</u> Determine the surface area of the solid obtained by rotating the curve (part of the astroid)

$$x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le \frac{\pi}{2}$$

about the x-axis.



$$S = \int_0^{\pi/2} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 2\pi \int_0^{\pi/2} \sin^3 t \sqrt{9\cos^4 t \sin^2 t + 9 \sin^4 t \cos^2 t}\, dt$$

$$= 2\pi \int_0^{\pi/2} \sin^3 t \,(3 \underbrace{\cos t \sin t}_{\ge 0 \text{ for } 0 < t \le \frac{\pi}{2}})\, dt$$

$$= 6\pi \int_0^{\pi/2} \sin^4 t \cos t \, dt = 6\pi \int_0^1 u^4 \, du = \boxed{\frac{6\pi}{5}}$$

$(u = \sin t, \; du = \cos t\, dt)$

(Also, read Example 9, p. 571)