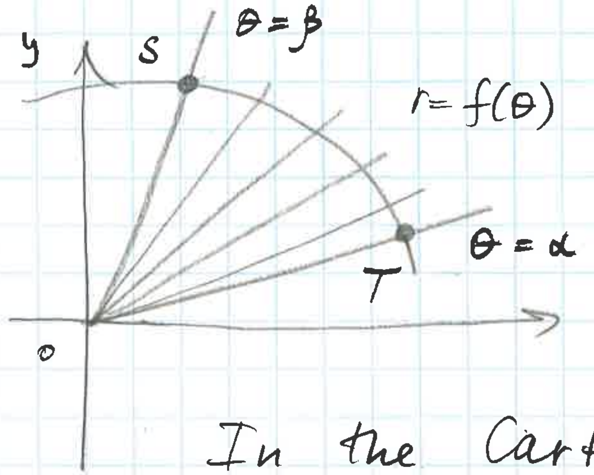
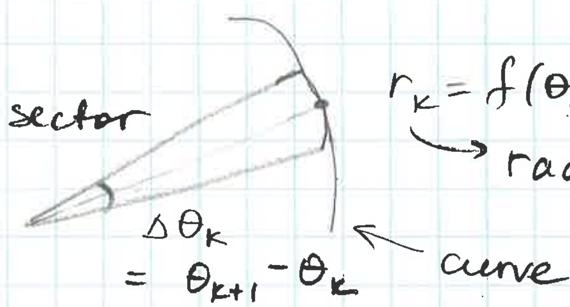
§10.5 Areas & lengths in Polar Coordinates.



Let us find area A of the region OTS bounded by the rays $\theta = \alpha$ & $\theta = \beta$ & the curve $r = f(\theta)$

In the Cartesian system we used rectangles to calculate areas. Here we will use "pie pieces":

Approximate the region OTS with n nonoverlapping fan-shaped circlular sectors based on a partition P of the angle $\angle SOT$:



$r_k = f(\theta_k)$ → radius of the sector

$$A_k = \underbrace{\left(\frac{\Delta\theta_k}{2\pi}\right)}_{\text{fraction of the circle}} \underbrace{\pi r_k^2}_{\text{area of the circle}}$$

$$\Rightarrow A_k = \frac{\Delta\theta_k \, r_k^2}{2} = \frac{1}{2} f(\theta_k)^2 \Delta\theta_k \Rightarrow$$

Total area $A \approx \sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2} f(\theta_k)^2 \Delta\theta_k$.

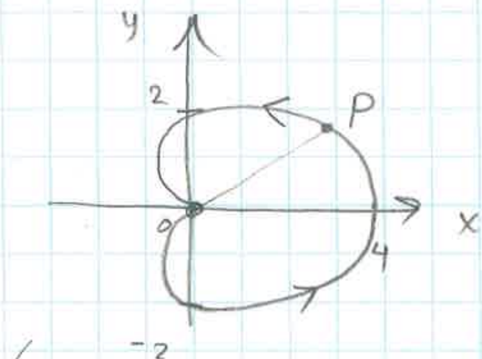
Take limit as $\Delta\theta_k \to 0$ ($n \to \infty$) to obtain

$$\lim_{n\to\infty} \sum_{k=1}^{n} A_k = \int_\alpha^\beta \frac{1}{2} f(\theta)^2 \, d\theta$$

So, $\boxed{A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta}$

Example 1

Find the area of the region in the xy-plane enclosed by the cardioid $r = 2(1+\cos\theta)$.



(symmetry about the x-axis)

As θ increases from 0 to 2π, the radius OP sweeps the region exactly once.

θ	r
0	4
$\pi/3$	3
$\pi/2$	2
$2\pi/3$	1
π	0

So, $A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} \cdot 4(1+\cos\theta)^2 \, d\theta$

$= \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta) \, d\theta = \int_0^{2\pi} (2 + 4\cos\theta + 1 + \cos 2\theta) \, d\theta$

$= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta) \, d\theta = \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$

$= 6\pi - 0 = \boxed{6\pi}$

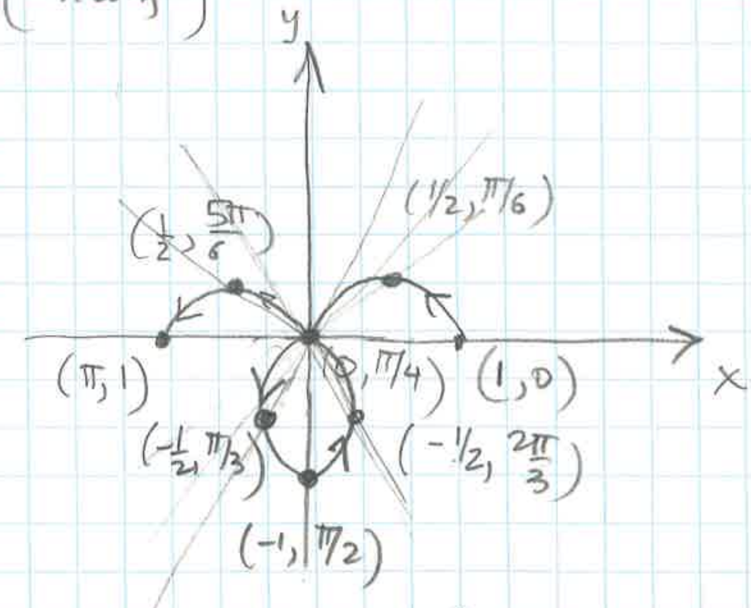
Example 2

Find the area of the region inside one leaf of the curve $r = \cos 2\theta$

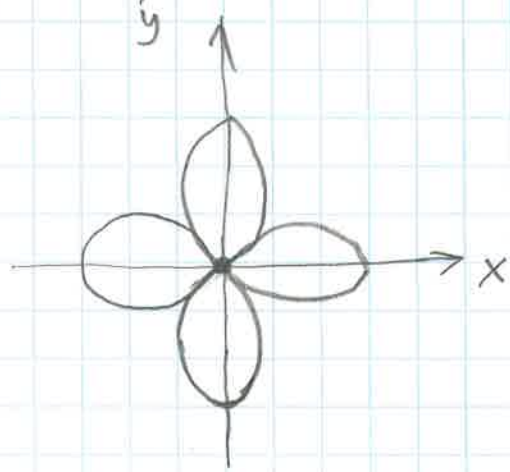
Note: symmetry about the x-axis ($\cos\theta$ is an even function)

θ	$r=\cos 2\theta$
0	1
$\pi/6$	$1/2$
$\pi/4$	0
$\pi/3$	$-1/2$
$\pi/2$	-1
$2\pi/3$	$-1/2$
$3\pi/4$	0
$5\pi/6$	$1/2$
π	1

Curve for $0 \le \theta \le \pi$ (half)



When you reflect the curve for $0 \le \theta \le \pi$ across the x-axis, you'll get the complete curve for $0 \le \theta \le 2\pi$:



"Four-leaved rose"

(you saw it: Exer. 25 in WeBWork 1)

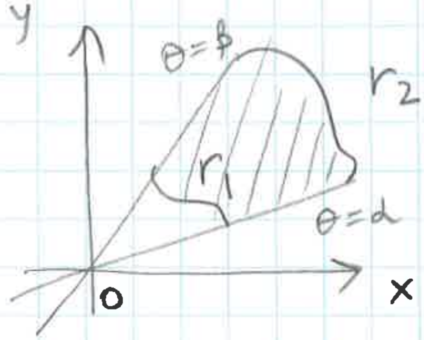
So, the area of one leaf is

$$A = \int_{\pi/4}^{3\pi/4} \frac{1}{2}(\cos 2\theta)^2 \, d\theta = \frac{1}{2}\int_{\pi/4}^{3\pi/4} \frac{1+\cos 4\theta}{2} \, d\theta$$

$$= \frac{1}{4}\left(\theta + \frac{1}{4}\sin 4\theta\right)\Bigg|_{\pi/4}^{3\pi/4} = \frac{1}{4}\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) = \boxed{\frac{\pi}{8}}$$

Area of the region $0 \le r_1(\theta) \le r_2(\theta)$,

$\alpha \le \theta \le \beta$

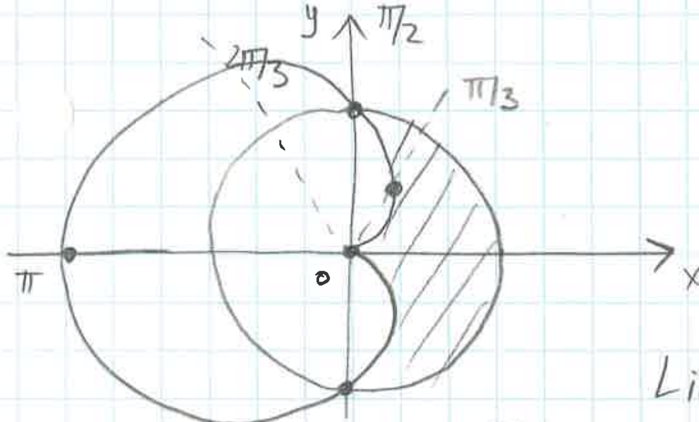


$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 \, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 \, d\theta =$$

$$= \int_{\alpha}^{\beta} \frac{1}{2}(r_2^2 - r_1^2)\, d\theta$$

Example 3. Find the area of the region that lies inside the circle $r=1$ and outside the cardioid $r = 1-\cos\theta$.

Sketch the region:



θ	$r_1 = 1-\cos\theta$
0	0
$\pi/3$	$1/2$
$\pi/2$	1
$2\pi/3$	$3/2$
π	2

$r_2 = 1$ – circle of radius 1

Limits of integration: $\theta = -\frac{\pi}{2}$ & $\theta = \pi/2$

So, $A = \displaystyle\int_{-\pi/2}^{\pi/2} \frac{1}{2}(r_2^2 - r_1^2)\, d\theta$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(1-(1-\cos\theta)^2\right) d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (2\cos\theta - \cos^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(2\cos\theta - \frac{1+\cos 2\theta}{2}\right) d\theta = \frac{1}{2}\left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\left(2-\frac{\pi}{4}\right) - \left(-2-\frac{\pi}{4}\right)\right] = \boxed{2-\frac{\pi}{4}}$$

Length of a Polar Curve.

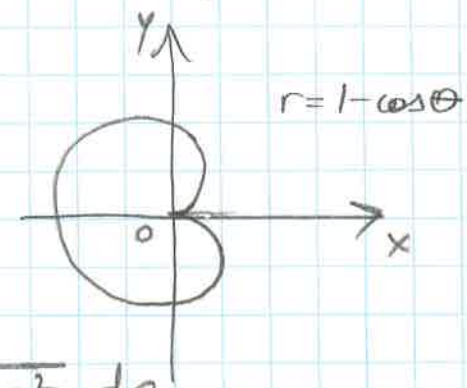
If $r = f(\theta)$ $\Rightarrow$ $x = r\cos\theta = f(\theta)\cos\theta$
$\alpha \le \theta \le \beta$ $\quad y = r\sin\theta = f(\theta)\sin\theta$

Then from §10.2, the length is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

(differentiate x & y & simplify to see)

<u>Example 4</u>: Find the length of the cardioid $r = 1 - \cos\theta$.



$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta = \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta$$

$$= \int_0^{2\pi} \left|2\sin\frac{\theta}{2}\right| d\theta = \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta$$

$$\left[0 \le \frac{\theta}{2} \le \pi \Rightarrow \sin\frac{\theta}{2} \geq 0\right]$$

$$= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = 4 + 4 = \boxed{8}$$