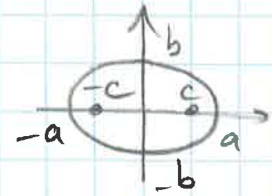
§10.6 Conics in Polar Coordinates.

(Used in astronomy & astronautical engineering)

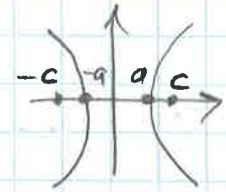
- **Eccentricity e:** (reveals conic's type)

(1) Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad (a > b)$;



$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}$$

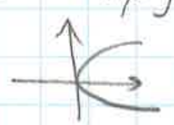
(2) Hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$;



$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} \qquad (b^2 = c^2 - a^2)$$

(3) Parabola : $(x^2 = 4py$ or $y^2 = 4px)$

$$e = 1.$$



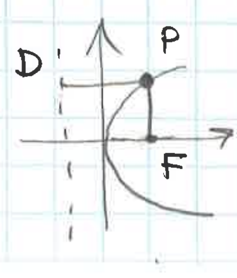
- For (1) & (2), $e = \dfrac{\text{distance between foci}}{\text{distance between vertices}}$

- Focus-directrix equation : $\boxed{PF = e \cdot PD}$

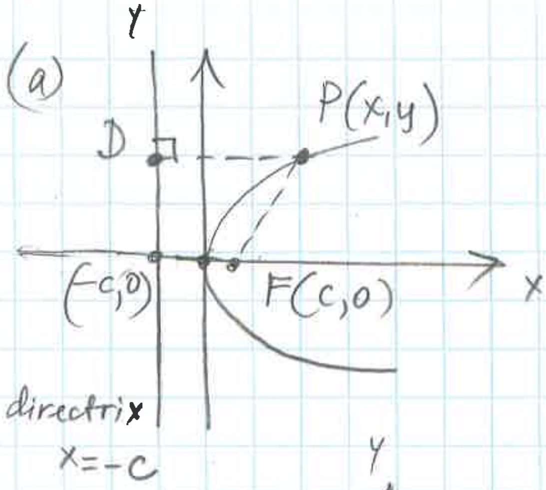
If PF is the distance from a point P on a conic curve to a fixed point F (focus), and PD is the distance from P to a fixed line (directrix), and

$PF = e \cdot PD$, then the path traced by P is

(a) a parabola if $e = 1$
(b) an ellipse if $e < 1$
(c) a hyperbola if $e > 1$.

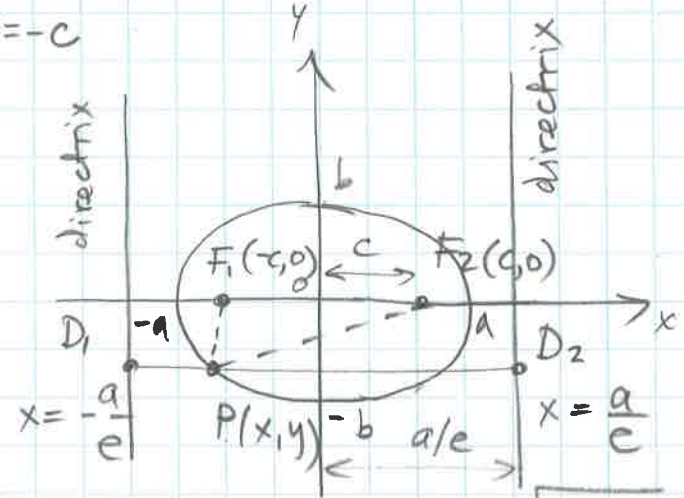


(a)



$$\frac{PF}{PD} = 1 \quad \text{(parabola)}$$

(b)

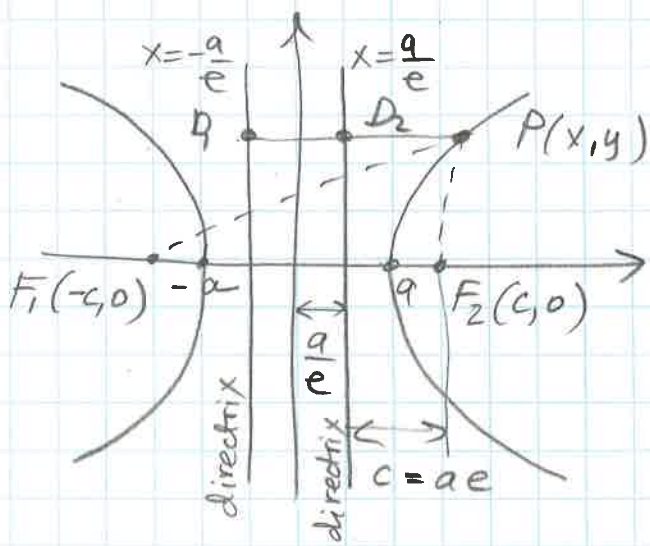


$$\frac{PF_1}{PD_1} = \frac{PF_2}{PD_2} = e < 1$$

(ellipse)

$$\left(e = c/a = \frac{\sqrt{a^2-b^2}}{a} \right)$$

(c)

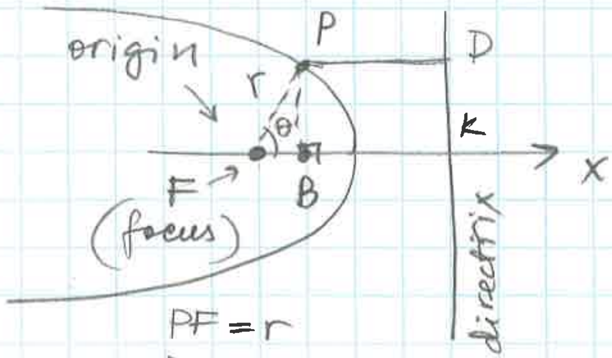


$$\frac{PF_1}{PD_1} = \frac{PF_2}{PD_2} = e > 1$$

(hyperbola)

Polar Equations:

We place one focus at the origin & the corresponding directrix to the right of the origin, along $x = k$



$PF = r$
$FB = r\cos\theta$ $\quad x = k$

$PF = r$, $FB = r\cos\theta$
$PD = k - FB = k - r\cos\theta$

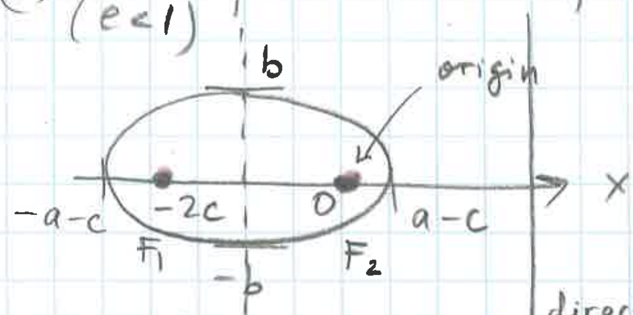
$\Rightarrow$ $PF = e \cdot PD$ becomes

$r = e(k - r\cos\theta)$ or

$$\boxed{r = \frac{ke}{1 + e\cos\theta}}$$ (solved for r)

$x = k > 0$ is the vertical directrix

(1) Ellipse $(e<1)$: shift focus to origin



$$\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = 1$$

$x = \frac{a}{e} - c = \frac{b^2}{c}$

$\left(e = \frac{c}{a}\right)$

$c^2 = a^2 - b^2$

$\Rightarrow$ Polar:

$$r = \frac{b^2/c \cdot c/a}{1 + \frac{c}{a}\cos\theta}$$

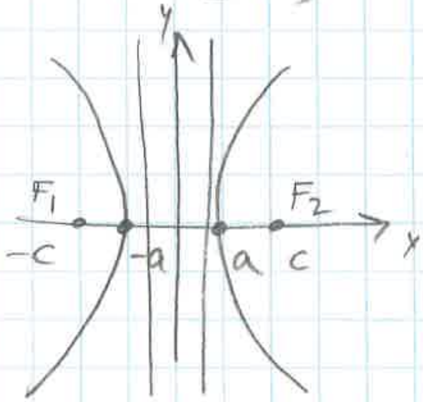
or

$$\boxed{r = \frac{a(1-e^2)}{1 + e\cos\theta}}$$

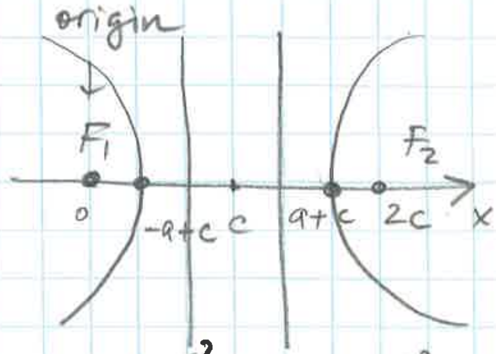
$a > b$

$\left(\text{or } r = \frac{a(1-e^2)}{1 + e\sin\theta} \right.$ if the major axis is along y-axis, that is directrix is $\left. y = k \right)$
$b > a$

(2) Hyperbola: ($e>1$) shift focus to origin



move focus →



$x = \pm \frac{a}{e} = \pm \frac{a^2}{c}$

$x = c - \frac{a^2}{c} = b^2/c$ $\quad x = c + \frac{a^2}{c}$

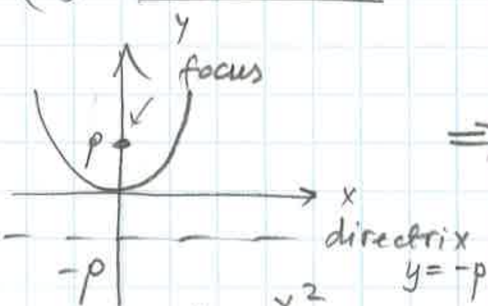
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Polar: $r = \dfrac{\frac{b^2}{c} \cdot \frac{c}{a}}{1 + \frac{c}{a}\cos\theta}$

$c^2 = a^2 + b^2$

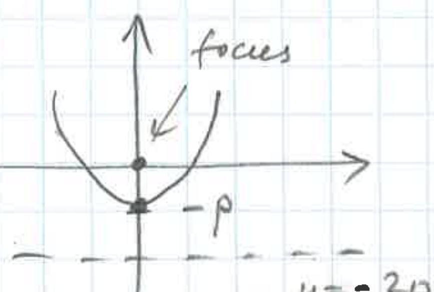
or $\boxed{r = \dfrac{a(e^2-1)}{1+e\cos\theta}}$ (or w/ $\sin\theta$)

(3) Parabola: $e=1$, $PF = PD$



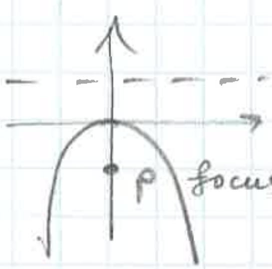
$y = \frac{x^2}{4p}$

⟹

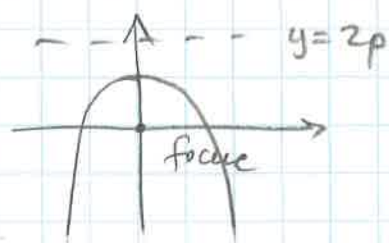


$y = -2p$

$r = -\dfrac{2p}{1+\sin\theta}$



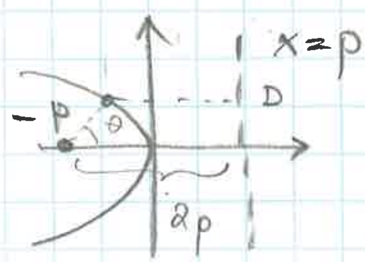
$y = p$ ⟹



$y = 2p$

$r = \dfrac{2p}{1+\sin\theta}$

shift ⟹ $r = -\dfrac{2p}{1+\cos\theta}$

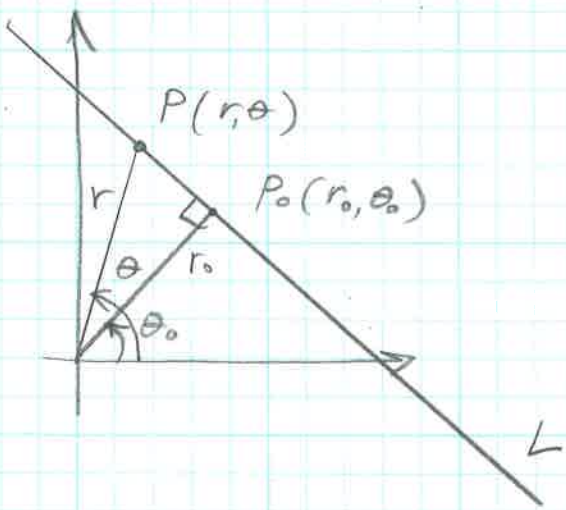


$x=p$ shift ⟹ $r = \dfrac{2p}{1+\cos\theta}$

Notes:

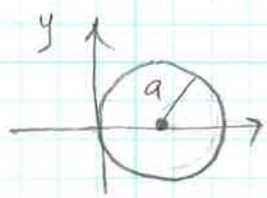
1) Lines: given by polar equations

$$r\cos(\theta-\theta_0)=r_0$$

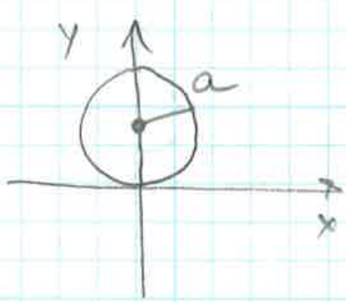


2) Circles:

$r = 2a\cos\theta$ → circle of radius a centered at a point on the pos. x-axis



$r = 2a\sin\theta$ → circle of radius a centered at a point on the pos. y-axis



See more on pages 590-591.