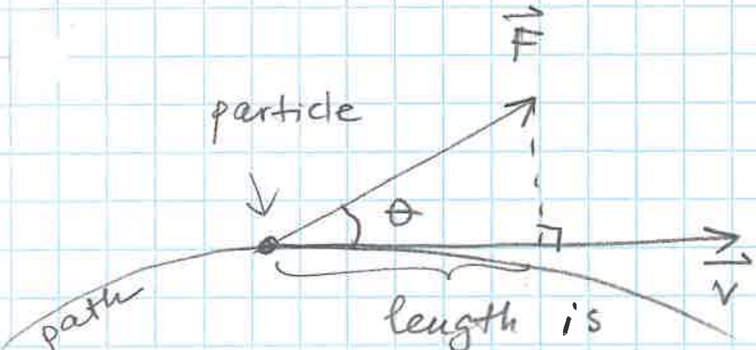
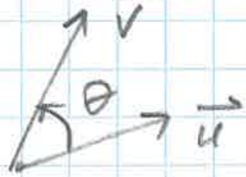
§ 11.3 The Dot Product.

$\underbrace{\text{Dot Product}}$ also inner product or scalar product



$|F|\cos\theta \rightarrow$ magnitude of $\vec{F}$ in the direction of motion

- Angle Between Vectors:



Thm. The angle θ between nonzero $\vec{u} = \langle u_1, u_2, u_3 \rangle$ & $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\vec{u}||\vec{v}|} \right) \quad (*)$$

"$\vec{u}$ dot $\vec{v}$"

- Def: the dot product $\vec{u}\cdot\vec{v}$ (also called inner or scalar product) of $\vec{u} = \langle u_1, u_2, u_3 \rangle$ & $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is the scalar $\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$.

- Example 1: $\vec{u} = \langle 1, 3, -5 \rangle$, $\vec{v} = \langle 4, -2, -1 \rangle \Rightarrow$

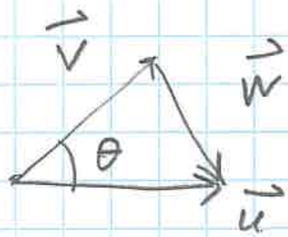
$$\vec{u}\cdot\vec{v} = 1\cdot 4 + 3\cdot(-2) + (-5)(-1) = 4 - 6 + 5 = \boxed{3}$$

(Using $\vec{i}, \vec{j}, \vec{k}$-notation for $\vec{u} = \langle \frac{1}{2}, 3, 1 \rangle$, $\vec{v} = \langle 4, -1, 2 \rangle$:

$$\left(\tfrac{1}{2}\vec{i} + 3\vec{j} + \vec{k}\right)\cdot\left(4\vec{i} - \vec{j} + 2\vec{k}\right) = \left(\tfrac{1}{2}\right)(4) + 3(-1) + 1\cdot 2 = \boxed{1}$$

)

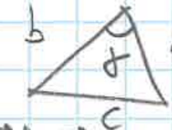
(*) Exploration: Consider $\vec{u}, \vec{v}, \vec{w}$:



$$\vec{u} = \vec{v} + \vec{w} \Rightarrow \vec{w} = \vec{u} - \vec{v}$$

$$\vec{w} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$$

$\Rightarrow$ using the Law of cosines

($c^2 = a^2 + b^2 - 2ab\cos\gamma$ ) :

$$|\vec{w}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta \Rightarrow$$

$$2|\vec{u}||\vec{v}|\cos\theta = |\vec{u}|^2 + |\vec{v}|^2 - |\vec{w}|^2$$

$$= \underbrace{u_1^2 + u_2^2 + u_3^2}_{|\vec{u}|^2} + \underbrace{v_1^2 + v_2^2 + v_3^2}_{|\vec{v}|^2} - \underbrace{\left((u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2\right)}_{|\vec{w}|^2} = 2(u_1 v_1 + u_2 v_2 + u_3 v_3)$$

$$= 2\,\vec{u}\cdot\vec{v} \Rightarrow \boxed{\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}}$$

Example 2:

Find the angle between $\vec{u} = 2\vec{i} + 10\vec{j} - 11\vec{k}$ & $\vec{v} = 2\vec{i} + 2\vec{j} + \vec{k}$.

Solution:

$$|\vec{u}| = \sqrt{2^2 + 10^2 + (-11)^2} = 15, \quad |\vec{v}| = \sqrt{2^2 + 2^2 + 1^2} = 3,$$

$$\vec{u}\cdot\vec{v} = 2\cdot 2 + 10\cdot 2 + (-11)\cdot 1 = 13$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{13}{15\cdot 3}\right) = \cos^{-1}\left(\frac{13}{45}\right) \approx 1.28 \text{ rad.}$$

- Orthogonal (Perpendicular) Vectors:

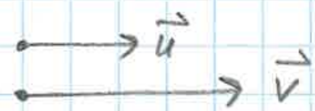
$\vec{u} \perp \vec{v}$ if $\theta = \frac{\pi}{2}$

$\Rightarrow \cos\theta = 0 \Rightarrow \vec{u}\cdot\vec{v} = 0$

So, $\boxed{\vec{u} \ \& \ \vec{v} \text{ are orthogonal iff } \vec{u}\cdot\vec{v} = 0}$

Q: How can we tell if $\vec{u} \parallel \vec{v}$?

$\theta = 0$ or π



$\theta = 0 \Rightarrow$

$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|$ $(\cos 0 = 1)$

$\theta = \pi \Rightarrow \vec{u}\cdot\vec{v} = -|\vec{u}||\vec{v}|$ $(\cos\pi = -1)$

Note: $\vec{0} \perp$ any $\vec{u}$ as $\vec{0}\cdot\vec{u} = 0u_1 + 0u_2 + 0u_3 = 0$

- Properties of $\vec{u}\cdot\vec{v}$: for any $\vec{u}, \vec{v}, \vec{w}$ and c - scalar,

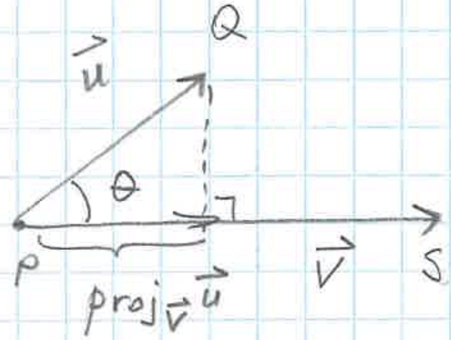
1) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
2) $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$
3) $(c\vec{u})\cdot\vec{v} = c(\vec{u}\cdot\vec{v}) = \vec{u}\cdot(c\vec{v})$
4) $\vec{u}\cdot\vec{u} = |\vec{u}|^2$
5) $\vec{0}\cdot\vec{u} = 0$

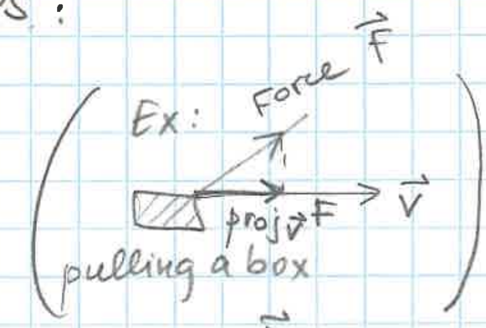
$\vec{u}\cdot\vec{u} = u_1u_1 + u_2u_2 + u_3u_3 = u_1^2 + u_2^2 + u_3^2 = |\vec{u}|^2$

(See other proofs on p. 613 or DIY!)

Vector Projection

Def: The vector projection of $\vec{u} = \vec{PQ}$ onto a nonzero $\vec{v} = \vec{PS}$ is the vector $\mathrm{proj}_{\vec{v}}\,\vec{u}$ determined by dropping a perpendicular from Q to the vector $\vec{PS}$:



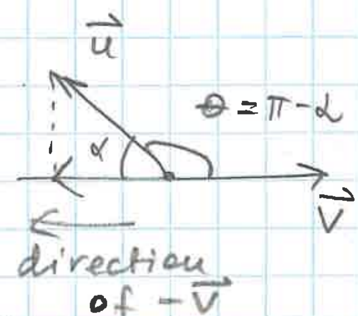


$\mathrm{proj}_{\vec{v}}\,\vec{F}$ is the effective force in the direction of $\vec{v}$

If $\theta < \frac{\pi}{2} \Rightarrow$

$$\mathrm{proj}_{\vec{v}}\,\vec{u} = \underbrace{(|u| \cos\theta)}_{\text{length}} \underbrace{\frac{\vec{v}}{|\vec{v}|}}_{\text{direction}}$$

If $\frac{\pi}{2} < \theta < \pi \Rightarrow$



$$\mathrm{proj}_{\vec{v}}\,\vec{u} = (|\vec{u}| \cos\alpha)\left(-\frac{\vec{v}}{|\vec{v}|}\right)$$

$$= (|\vec{u}| \cos(\pi - \theta))\left(-\frac{\vec{v}}{|\vec{v}|}\right)$$

$$= \underbrace{(-|\vec{u}| \cos\theta)}_{\text{length}}\underbrace{\left(-\frac{\vec{v}}{|\vec{v}|}\right)}_{\text{direction}} = (|\vec{u}| \cos\theta)\left(\frac{\vec{v}}{|\vec{v}|}\right)$$

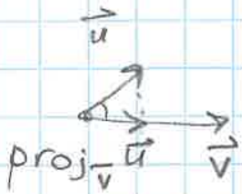
In either case:

$$\mathrm{proj}_{\vec{v}}\,\vec{u} = (|\vec{u}| \cos\theta)\left(\frac{\vec{v}}{|\vec{v}|}\right) = \left(|\vec{u}| \cdot \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}\right)\left(\frac{\vec{v}}{|\vec{v}|}\right)$$

$$= \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}\right)\left(\frac{\vec{v}}{|\vec{v}|}\right) = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right)\vec{v}.$$

- Summary:

$$\text{proj}_{\vec{v}}\vec{u} = \underbrace{\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\right)}_{\text{scalar component of } \vec{u} \text{ in the direction of } \vec{v}}\underbrace{\left(\frac{\vec{v}}{|\vec{v}|}\right)} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}$$



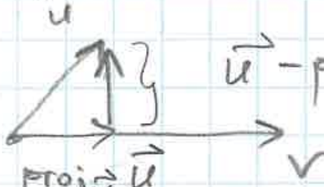
Example 3: Find $\text{proj}_{\vec{v}}\vec{u}$ given $\vec{u} = \sqrt{2}\,\vec{i} + \sqrt{3}\,\vec{j} + 2\vec{k}$ & $\vec{v} = -\vec{i} + \vec{j}$ & the scalar component of $\vec{u}$ in the direction of $\vec{v}$.

Sol: $$\text{proj}_{\vec{v}}\vec{u} = \underbrace{\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\right)}_{\text{scalar component}}\left(\frac{\vec{v}}{|\vec{v}|}\right)$$

$$= \frac{\sqrt{2}\cdot(-1) + \sqrt{3}\cdot 1 + 2\cdot 0}{\sqrt{(-1)^2 + (1)^2 + 0^2}} \cdot \left(\frac{-\vec{i} + \vec{j}}{\sqrt{(-1)^2 + 1^2 + 0^2}}\right)$$

$$= \underbrace{\frac{-\sqrt{2} + \sqrt{3}}{\sqrt{2}}}_{\substack{\text{scalar component in the dir. of } \vec{v}, \\ \text{here it's also } |\text{proj}_{\vec{v}}\vec{u}|}} \cdot \frac{-\vec{i} + \vec{j}}{\sqrt{2}} = \left(\frac{\sqrt{2} - \sqrt{3}}{2}\right)\vec{i} + \left(\frac{-\sqrt{2} + \sqrt{3}}{2}\right)\vec{j}$$

- Writing a vector as a sum of orthogonal vectors:

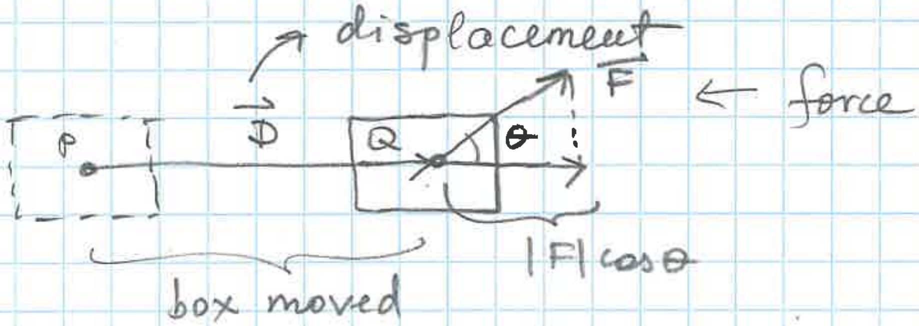


$\vec{u} - \text{proj}_{\vec{v}}\vec{u} \perp \text{proj}_{\vec{v}}\vec{u}$ (See Example 6 pp. 614-615)

$$\vec{u} = \text{proj}_{\vec{v}}\vec{u} + \left(\vec{u} - \text{proj}_{\vec{v}}\vec{u}\right)$$

$$= \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v} + \left(\vec{u} - \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}\right)$$

Work: (done by a force)



$\vec{D} = \vec{PQ}$

Work $W = \underbrace{(|\vec{F}| \cos\theta)}_{\text{scalar component in the direction of displacement } \vec{D}} (\text{length of } \vec{D})$

done by $\vec{F}$ to cause a displacement $\vec{D}$ of the object

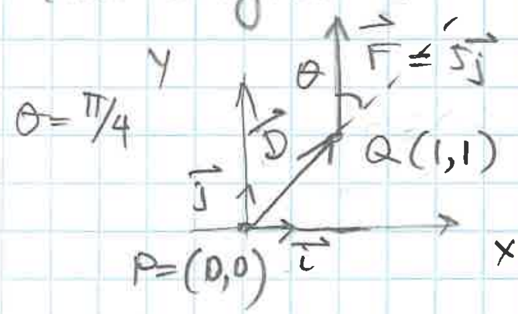
$$= |\vec{F}| \cos\theta \cdot |\vec{D}| = |\vec{F}| \left(\frac{\vec{F} \cdot \vec{D}}{|\vec{F}||\vec{D}|} \right) |\vec{D}| = \vec{F} \cdot \vec{D}$$

So, $\boxed{W = \vec{F} \cdot \vec{D}}$

(See Example 7, p. 615)

Example 4.

Find the work done by a force $\vec{F} = 5\vec{j}$, in moving an object along the line from the origin to the point $(1,1)$



$|\vec{F}| = 5\,N$ (5 newtons) $\quad (1N = 1 kg \cdot m/s^2)$

$\vec{D} = \langle 1,1 \rangle \Rightarrow |\vec{D}| = \sqrt{2}\, m$

Thus, $W = \vec{F} \cdot \vec{D} = |\vec{F}| \cdot |\vec{D}| \cdot \cos\theta$

$$= 5 \cdot \sqrt{2} \cdot \cos\frac{\pi}{4} = 5 \cdot \sqrt{2} \cdot \frac{\sqrt{2}}{2} = 5\,Nm = 5\,J \text{ (joules)}$$

$(\vec{F} \cdot \vec{D} = \langle 5,0 \rangle \cdot \langle 1,1 \rangle = 5\,J)$