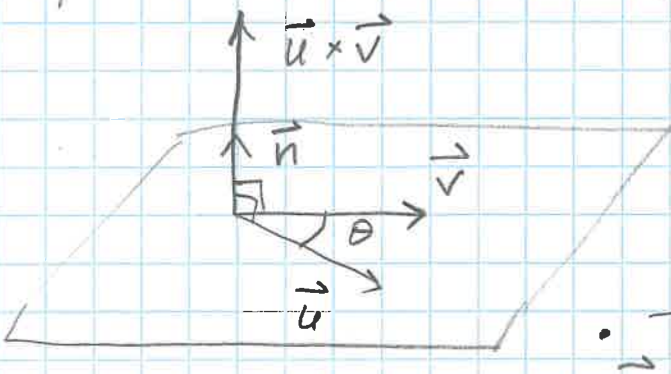
§ 11.4 The Cross (or the Vector) Product.

3D space: to describe how a plane is tilting, we introduce the vector or cross product
(2D: slope described how a line is tilting)



- Start with nonzero $\vec{u}$ & $\vec{v}$, if $\vec{u} \not\parallel \vec{v}$, they define a plane.
- Then select a unit vector $\vec{n} \perp \vec{u}$ & $\vec{v}$ (plane) by the right-hand rule.

RH rule: $\vec{n}$ points the way your thumb points when your fingers curl from $\vec{u}$ to $\vec{v}$ (Through the angle θ).

Def: Cross (Vector) Product (only applies in 3D)

$$\vec{u} \times \vec{v} = \big(\underbrace{|\vec{u}||\vec{v}| \sin\theta}_{\text{length}}\big) \underbrace{\vec{n}}_{\text{direction}}$$

("$\vec{u}$ cross $\vec{v}$")

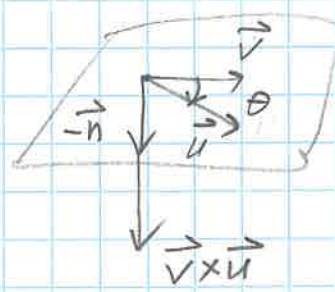
- $\vec{u} \times \vec{v}$ is a vector (vs. $\vec{u} \cdot \vec{v}$, a scalar)
- $(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$ & $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$ (as $\vec{u} \times \vec{v} \perp \vec{u}$ & $\vec{v}$)
- $\vec{u} \parallel \vec{v} \iff \vec{u} \times \vec{v} = 0$ ($\sin 0 = \sin \pi = 0$)

Properties of $\vec{u} \times \vec{v}$:

$\vec{u}, \vec{v}, \vec{w}$ - vectors; r, s - scalars

(1) $(r\vec{u}) \times (s\vec{v}) = (rs)(\vec{u} \times \vec{v})$

(2) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$



(3) $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$

(4) $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$

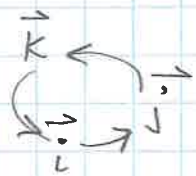
(5) $\vec{0} \times \vec{u} = \vec{0}$

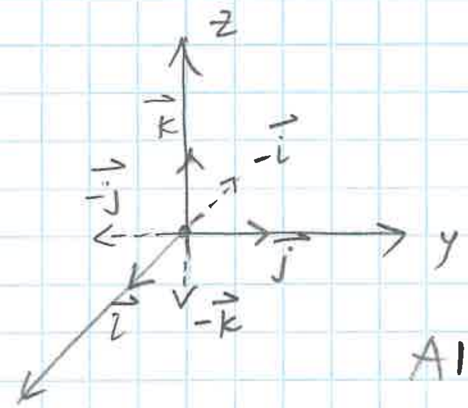
(6) $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

Also: $(\vec{u} \times \vec{v}) \times \vec{w} \neq \vec{u} \times (\vec{v} \times \vec{w})$

[Proofs by definition (p. 619)]

<u>$\vec{i}, \vec{j}, \vec{k}$</u> :





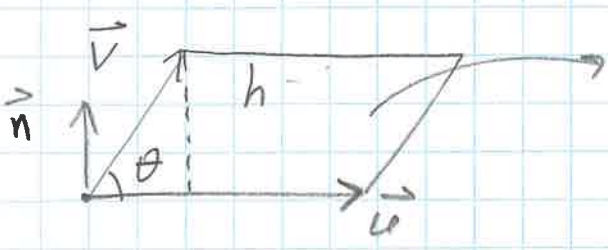
$\vec{i} \times \vec{j} = -(\vec{j} \times \vec{i}) = \vec{k}$

$\vec{j} \times \vec{k} = -(\vec{k} \times \vec{j}) = \vec{i}$

$\vec{k} \times \vec{i} = -(\vec{i} \times \vec{k}) = \vec{j}$

Also: $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$

<u>Area of a Parallelogram:</u>



$$\text{Area} = \text{base} \cdot \text{height}$$
$$= |\vec{u}| \cdot \underbrace{|\vec{v}| \sin\theta}_{h}$$
$$= |\vec{u} \times \vec{v}|$$

$\left(\text{Since } |\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}||\sin\theta| \underbrace{|\vec{n}|}_{=1} = |\vec{u}||\vec{v}| \sin\theta \right)$

$\sin\theta \geq 0$

<u>Determinant Formula for $\vec{u} \times \vec{v}$:</u>

If $\vec{u} = \langle u_1, u_2, u_3 \rangle = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ &
$\vec{v} = \langle v_1, v_2, v_3 \rangle = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$, then

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

See p. 619

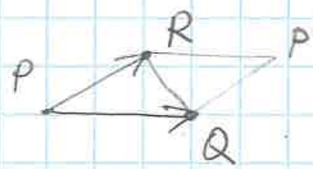
$$\left(\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2)\vec{i} - (u_1 v_3 - u_3 v_1)\vec{j} + (u_1 v_2 - u_2 v_1)\vec{k} \right)$$

Recall:

$$\left(\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right)$$

Example 1:

Find the area of a triangle PQR with $P(-2,2,0)$, $Q(0,1,-1)$, $R(-1,2,-2)$.



Area of ΔPQR is $1/2$ area of parallelogram $PRP'Q$, i.e.

$$\frac{1}{2}|\vec{PQ} \times \vec{PR}|$$

Find

$$\vec{PQ} \times \vec{PR} = \langle 2, -1, -1 \rangle \times \langle 1, 0, -2 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & -1 \\ 1 & 0 & -2 \end{vmatrix}$$

$$= \left((-1)(-2) - (-1)\cdot 0\right)\vec{i} - \left(2(-2) - (-1)(1)\right)\vec{j} + \left(2\cdot 0 - (-1)1\right)\vec{k}$$

$$= 2\vec{i} + 3\vec{j} + \vec{k} \implies$$

$$\text{Area of } \Delta PQR = \frac{1}{2}|\langle 2,3,1 \rangle| = \frac{1}{2}\sqrt{4+9+1}$$

$$= \boxed{\sqrt{14}/2}$$

Read Examples 1, 2 on p. 620.

Example 2. Find a unit vector $\perp$ to the plane of $P(-2,2,0)$, $Q(0,1,-1)$, and $R(-1,2-2)$

$\overrightarrow{PQ} \times \overrightarrow{PR} \perp$ plane. In Example 1,

$\overrightarrow{PQ} \times \overrightarrow{PR} = 2\vec{i} + 3\vec{j} + \vec{k} \Rightarrow$ a unit vector is

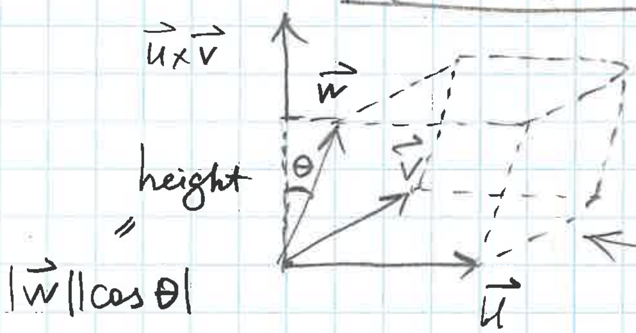
$$\vec{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \frac{2\vec{i} + 3\vec{j} + \vec{k}}{\sqrt{14}} = \frac{2}{\sqrt{14}}\vec{i} + \frac{3}{\sqrt{14}}\vec{j} + \frac{1}{\sqrt{14}}\vec{k}$$

- Note that $-\vec{n}$ also satisfies the condition.

→ Read about <u>Torque</u> on page 621.

- <u>Triple Scalar or Box Product</u> $\underbrace{\text{of } \vec{u}, \vec{v}, \vec{w}}_{\text{order matters!}}$

is $\boxed{(\vec{u} \times \vec{v}) \cdot \vec{w}}$ (scalar!)



Parallelepiped determined by $\vec{u}, \vec{v}, \vec{w}$:

base has area $|\vec{u} \times \vec{v}|$

So, the volume is $|\vec{u} \times \vec{v}||\vec{w}||\cos\theta|$

$$= \underbrace{\left|(\vec{u} \times \vec{v}) \cdot \vec{w}\right|}_{\text{abs. value}}$$

$\left(\text{Recall: } \cos\theta = \dfrac{(\vec{u} \times \vec{v}) \cdot \vec{w}}{|\vec{u} \times \vec{v}||\vec{w}|}\right)$

Can show: $(\vec{u} \times \vec{v}) \cdot \vec{w} = (\vec{v} \times \vec{w}) \cdot \vec{u} = (\vec{w} \times \vec{u}) \cdot \vec{v}$

Also: $(\vec{u} \times \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \times \vec{w})$

To compute $(\vec{u} \times \vec{v}) \cdot \vec{w}$:

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \cdot \vec{w}$$

$$= w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} =$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \overset{\text{or}}{=} u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

<u>Example 3:</u> Find the volume of the parallelepiped (box) determined by $\vec{u} = \vec{i} + 2\vec{j} - \vec{k}$, $\vec{v} = -2\vec{i} + 3\vec{k}$, $\vec{w} = 7\vec{j} - 4\vec{k}$

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = (1)\left(0(-4) - 3 \cdot 7\right) - (2)\left(-2 \cdot (-4)\right.$$

$$\left. + 3 \cdot 0\right) - (1)\left((-2)7 - 0\right) = -21 - 16 + 14 = -23 \Rightarrow$$

The volume is $|(\vec{u} \times \vec{v}) \cdot \vec{w}| = |-23| = 23$ units^3.