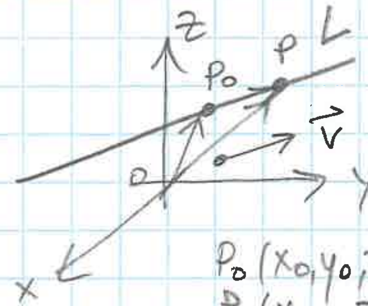
§ 11.5 Lines & Planes in Space.

- In the plane: point & slope determine a line.
- In space: point & vector determine a line.

$\downarrow$ gives the direction



$P_0(x_0, y_0, z_0)$
$P(x, y, z)$

If a line L in space is passing through the point $P_0(x_0, y_0, z_0)$, parallel to $\vec{V}$, then L is the set of all point $P(x, y, z)$ with $\overrightarrow{P_0P} \parallel \vec{V}$.

$\Rightarrow$ $\overrightarrow{P_0P} = t\vec{V}$ for a scalar t that depends on the pt P.

$\overrightarrow{P_0P} = t\vec{v}$, i.e.

$$(x-x_0)\vec{i} + (y-y_0)\vec{j} + (z-z_0)\vec{k} = t\left(v_1\vec{i} + v_2\vec{j} + v_3\vec{k}\right)$$

$$\Rightarrow \quad x\vec{i} + y\vec{j} + z\vec{k} = (x_0 + tv_1)\vec{i} + (y_0 + tv_2)\vec{j} + (z_0 + tv_3)\vec{k}$$

$$\Rightarrow \quad \begin{aligned} x &= x_0 + tv_1 \\ y &= y_0 + tv_2 \\ z &= z_0 + tv_3 \end{aligned}$$

if $\vec{r} = \langle x, y, z \rangle$ & $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$,

then $\boxed{\vec{r}(t) = \vec{r_0} + t\vec{v}, \quad -\infty < t < \infty}$ is

a vector equation for the line L through $P_0(x_0, y_0, z_0)$, parallel to $\vec{v}$.

($\vec{r}$ is the position vector of $P(x, y, z)$ & $\vec{r_0}$ is the position vector of $\underbrace{P_0(x_0, y_0, z_0)}_{\text{base point}}$ on the line L.)

Parametric Equations for a Line:

$$\underbrace{x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_2}_{\text{Standard parameterization}}, \quad -\infty < t < \infty$$

Example 1

Find a vector equation & parametric equations for the line that passes through the pt. $(5,1,3)$ and is parallel to $\vec{i} + 4\vec{j} - 2\vec{k}$.

$\vec{r}_0 = 5\vec{i} + \vec{j} + 3\vec{k}$ ← position vector of $P_0(5,1,3)$

$\vec{v} = \vec{i} + 4\vec{j} - 2\vec{k}$ ← vector ∥ line

⇒ so, $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ (vector equation) is

$$\vec{r}(t) = (5\vec{i} + \vec{j} + 3\vec{k}) + t(\vec{i} + 4\vec{j} - 2\vec{k}), \quad -\infty < t < \infty$$

The parametric equations are:

$$x = 5 + t, \quad y = 1 + 4t, \quad z = 3 - 2t, \quad -\infty < t < \infty$$

Example 2. Find the param. equations for the line through $P(-3,2,-3)$ & $Q(1,-1,4)$.

$$\vec{v} = \vec{PQ} = (1-(-3))\vec{i} + (-1-2)\vec{j} + (4-(-3))\vec{k}$$
$$= 4\vec{i} - 3\vec{j} + 7\vec{k}$$

the base pt. could be $\vec{P}$ or $\vec{Q}$. If we pick P:

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t.$$

(check what you get picking Q)

- Parameterizations are not unique!

e.g., $x = 1 + 4t, \; y = -1 - 3t, \; z = 4 + 7t$

parameterize the same line with $Q(1,-1,4)$

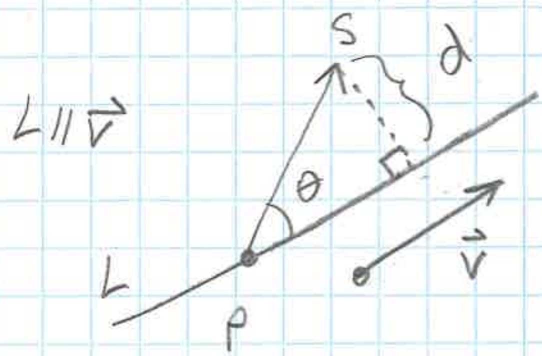
OR: $x = -3 + 4t^3, \; y = 2 - 3t^3, \; z = -3 + 7t^3$

- See Example 3 on parameterization of a segment of a line $\rightarrow$ need restriction on t! (p. 625)

- Note: $\vec{r}(t) = \vec{r}_0 + t\vec{v} = \vec{r}_0 + t|\vec{v}|\left(\dfrac{\vec{v}}{|\vec{v}|}\right) \rightarrow$ direction

  initial position, time, speed

- Distance from a Point to a Line in Space.



Distance $d = |\vec{PS}|\sin\theta$

Since $|\vec{PS}\times\vec{v}| = |\vec{PS}||\vec{v}|\sin\theta$

$\Rightarrow |\vec{PS}|\sin\theta = \dfrac{|\vec{PS}\times\vec{v}|}{|\vec{v}|}$

Thus, distance from a point S to a line through P, parallel to $\vec{v}$ is

$$\boxed{d = \frac{|\vec{PS}\times\vec{v}|}{|\vec{v}|}}$$

Example 3. Find the distance from $S(3,-1,4)$ to the line $L: x=-2+3t,\ y=-2t,\ z=1+4t$.

$L: \vec{v} = \langle 3,-2,4\rangle,\ P(-2,0,1)$ (when $t=0$)

$\vec{PS} = \langle 3-(-2), -1-0, 4-1\rangle = \langle 5,-1,3\rangle$

$$\vec{PS}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -1 & 3 \\ 3 & -2 & 4 \end{vmatrix} = ((-1)4-3(-2))\vec{i} - (5\cdot 4-3\cdot 3)\vec{j} + (5(-2)-(-1)\cdot 3)\vec{k} =$$

$= 2\vec{i} - 11\vec{j} - 7\vec{k}$

$|\vec{PS} \times \vec{v}| = \sqrt{2^2 + (-11)^2 + (-7)^2} = \sqrt{174}$

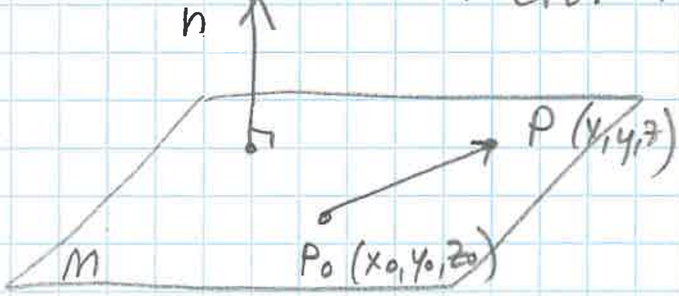
$|\vec{v}| = \sqrt{3^2 + (-2)^2 + 4^2} = \sqrt{29}$

So, $d = \dfrac{|\vec{PS} \times \vec{v}|}{|\vec{v}|} = \dfrac{\sqrt{174}}{\sqrt{29}} = \boxed{\sqrt{6}}$

- An Equation for a <u>Plane in Space.</u>

A point and a "tilt" determine a plane

"tilt" → vector $\perp$ plane



Consider plane M through a pt. $P_0(x_0,y_0,z_0)$ & normal ($\perp$) to $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$, $\vec{n} \neq \vec{0}$.

for any $P(x,y,z)$ on M, $\vec{P_0P} \perp \vec{n} \Rightarrow \vec{P_0P} \cdot \vec{n} = 0$

So, $\vec{P_0P} \cdot \vec{n} = \left((x-x_0)\vec{i} + (y-y_0)\vec{j} + (z-z_0)\vec{k}\right) \cdot (A\vec{i} + B\vec{j} + C\vec{k})$

$= A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$ → equation for M!

- The plane through $P_0(x_0,y_0,z_0)$, normal to $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$ has

  - vector equation: $\vec{n} \cdot \vec{P_0P} = 0$

  - Component equation: $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$

  - component equation simplified: $Ax + By + Cz = \underbrace{Ax_0 + By_0 + Cz_0}_{\text{number}}$

Example 4:

Find an equation of the plane that passes through the pts $P(1,3,2)$, $Q(3,-1,6)$, $R(5,2,0)$.

$\vec{n} \perp$ plane ?

$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$ where $\overrightarrow{PQ} = \langle 2,-4,4 \rangle$ and

$\overrightarrow{PR} = \langle 4,-1,-2 \rangle$

So, $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\vec{i} + 20\vec{j} + 14\vec{k}$

The component equation of the plane through $P(1,3,2)$ and perpendicular to $\vec{n}$,

is: $12(x-1) + 20(y-3) + 14(z-2) = 0$

Simplifying:

$$12x + 20y + 14z = 12 + 60 + 28$$

(component equation simplified!) $12x + 20y + 14z = 100$

Note that the components of $\vec{n}$ became the coefficients of x, y, z, i.e. the vector $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k} \perp$ plane $Ax + By + Cz = D$.

(Note: $12x + 20y + 14z = 100$ can be written as $6x + 10y + 7z = 50$ (divide by 2))

Lines of Intersection:

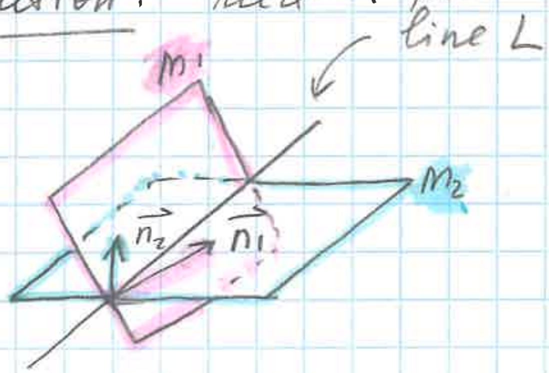
- Two planes are parallel if their normals are parallel, $\vec{n_1} \parallel \vec{n_2}$, i.e. $\vec{n_1} = k\vec{n_2}$, for some scalar k.

- Two non-parallel planes intersect in a line.

Example 5: Find the parametric equations for the line in which the planes

$M_1: x+y+z=1$ & $M_2: x-2y+3z=1$ intersect.

Solution: need a pt. & a vector:



1) $L \perp \vec{n_1}$ & $\vec{n_2} \Rightarrow$

$L \parallel \vec{n_1} \times \vec{n_2}$

$$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= \left(1\cdot 3 - 1(-2)\right)\vec{i} - \left(1\cdot 3 - 1\cdot 1\right)\vec{j} + \left(1(-2) - 1\cdot 1\right)\vec{k}$$

$$= 5\vec{i} - 2\vec{j} - 3\vec{k} \rightsquigarrow \text{vector} \parallel L.$$

2) Point on the line L? (Any pt common to M_1, M_2)

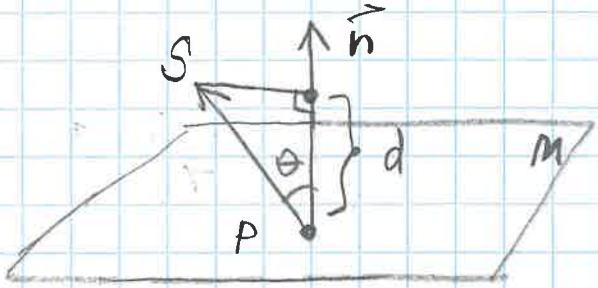
Let $z=0 \Rightarrow x+y=1$ & $x-2y=1 \rightarrow$ solve for x & y:

(arbitrary choice!)

$$\begin{array}{r} x+y=1 \\ - \quad x-2y=1 \\ \hline 3y=0 \end{array} \Rightarrow y=0 \Rightarrow x=1. \text{ So, } (1,0,0) \text{ is on } L.$$

3) L:
$$x = 1+5t$$
$$y = 0-2t$$
$$z = 0-3t$$

• The Distance from a Pt. to a Plane



$$d = \left| \text{proj}_{\vec{n}} \vec{PS} \right| = \left| \frac{\vec{PS} \cdot \vec{n}}{|\vec{n}|} \right|$$

length of projection of $\vec{PS}$ on $\vec{n}$.

(P is any pt on the plane)

(**Note:** $d = |\vec{PS}|\,|\cos\theta| = |\vec{PS}| \left| \frac{\vec{PS} \cdot \vec{n}}{|\vec{PS}||\vec{n}|} \right| = \left| \vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right|$)

Example 6: Find the distance from the pt. $S(1,5,-4)$ and the plane $3x - y + 2z = 6$.

Solution: $\vec{n} = \langle 3, -1, 2 \rangle$. Need a pt. on the plane:

pt. $P(2,0,0)$ will satisfy the equation $3x - y + 2z = 6$.

intercepts are easy!

$\vec{PS} = \langle -1, 5, -4 \rangle$, $|\vec{n}| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14} \Rightarrow$

$$d = \left| \frac{\vec{PS} \cdot \vec{n}}{|\vec{n}|} \right| = \left| \frac{(-1) \cdot 3 + 5(-1) + (-4) \cdot 2}{\sqrt{14}} \right| = \boxed{\frac{16}{\sqrt{14}}}$$

Angles Between Planes:

acute angle between the normals.

Example 7: Find the angle between

$M_1: x + y + z = 1$ & $M_2: x - 2y + 3z = 1$

Sol: $\vec{n_1} = \langle 1, 1, 1 \rangle$, $\vec{n}_2 = \langle 1, -2, 3 \rangle \Rightarrow \theta = \cos^{-1}\left(\frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|} \right)$

$$= \cos^{-1}\left(\frac{2}{\sqrt{3}\,\sqrt{14}} \right) = \cos^{-1}\left(\frac{2}{\sqrt{42}} \right) \approx 1.257 \text{ rad.}$$