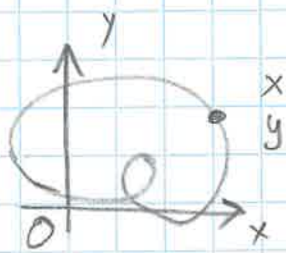
Chapter 12 Vector-Valued Functions & Motion in Space

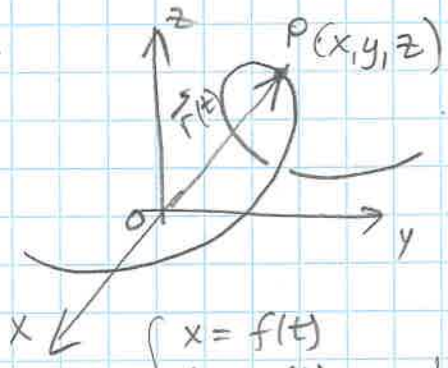
§12.1 Curves in Space & Their Tangents.

$\mathbb{R}^2$:



$\left.\begin{array}{l} x=f(t) \\ y=g(t) \end{array}\right\}$ param. equ's

$\mathbb{R}^3$:



particle path, curve $\left\{\begin{array}{l} x=f(t) \\ y=g(t) \\ z=h(t) \end{array}\right.$ $\quad t\in I$

<u>Def</u>: The vector $\overrightarrow{OP}=\vec{r}(t)=f(t)\vec{i}+g(t)\vec{j}+h(t)\vec{k}$ at time t is the particle <u>position vector</u>, w/ f,g,h component functions.

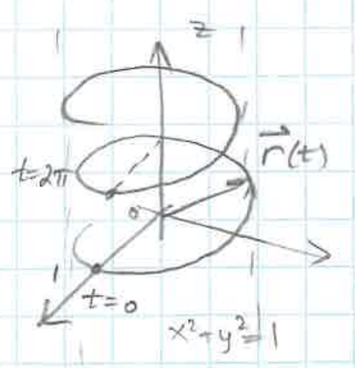
<u>Def</u>: A vector (or vector-valued) function is a function whose domain is a set of real numbers and whose range is a set of vectors.

$\vec{r}(t): \mathbb{R}\to\mathbb{R}^3$ (or some interval in $\mathbb{R}$) (Domain = common domain of components f,g,h)

(Real-valued or scalar functions : $y=f(x)$, $g(x,y)=z$)

<u>Example 1</u> Graph $\vec{r}(t)=\underbrace{(\cos t)}_{f(t)}\vec{i}+\underbrace{(\sin t)}_{g(t)}\vec{j}+\underbrace{t}_{h(t)}\vec{k}$

($\vec{r}(t)$ is defined for all $t\in\mathbb{R}$)



As t increases, pt. $(\cos t, \sin t, t)$ winds around the circular cylinder $x^2+y^2=1$ $(\cos^2 t+\sin^2 t=1)$. Periodicity results in repeating the curve's one turn each 2π-period. Called a <u>helix</u>

$x = \cos t,\ y = \sin t,\ z = t$ (helix = spiral in Greek)

$(-\infty < t < \infty)$ parametric equations for the helix.

- Limits & Continuity: (similar to scalar func's)

Def: Let $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ be a vector function w/ domain D, and $\vec{L}$ a vector.

We say (write) $\lim\limits_{t \to t_0} \vec{r}(t) = \vec{L}$ if for every $\varepsilon > 0$, there exists $\delta > 0$ $(\delta = \delta(\varepsilon))$ s.t. for all $t \in D$

$|\vec{r}(t) - \vec{L}| < \varepsilon$ whenever $0 < |t - t_0| < \delta$.

Given $\vec{L} = L_1\vec{i} + L_2\vec{j} + L_3\vec{k}$ and $\lim\limits_{t\to t_0} f(t) = L_1$, $\lim\limits_{t\to t_0} g(t) = L_2$, $\lim\limits_{t\to t_0} h(t) = L_3$

$$\lim_{t \to t_0} \vec{r}(t) = \underbrace{\left(\lim_{t\to t_0} f(t)\right)}_{L_1}\vec{i} + \underbrace{\left(\lim_{t\to t_0} g(t)\right)}_{L_2}\vec{j} + \underbrace{\left(\lim_{t\to t_0} h(t)\right)}_{L_3}\vec{k}$$

Example 2: $\vec{r}(t) = (1+t^3)\vec{i} + te^{-t}\vec{j} + \frac{\sin t}{t}\vec{k}$

$$\lim_{t \to 0} \vec{r}(t) = \left[\lim_{t\to 0}(1+t^3)\right]\vec{i} + \left[\lim_{t\to 0} te^{-t}\right]\vec{j} + \left[\lim_{t\to 0}\frac{\sin t}{t}\right]\vec{k}$$
$$= (1)\vec{i} + (0)\vec{j} + (1)\vec{k} = \vec{i} + \vec{k}.$$

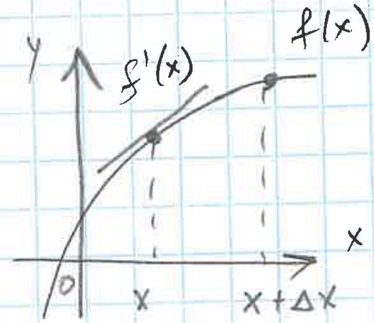
Def: A vector function $\vec{r}(t)$ is continuous at t_0 if $\lim\limits_{t\to t_0}\vec{r}(t) = \vec{r}(t_0)$.

Example 3: (helix) $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}$ is cont. for any $t \in \mathbb{R}$ since $\cos t, \sin t, t$ cont. everywhere

③

• Derivatives & Motion.

$\mathbb{R}^2$



$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$\mathbb{R}^3$: Let $\vec{\Delta r} = \vec{r}(t+\Delta t) - \vec{r}(t)$

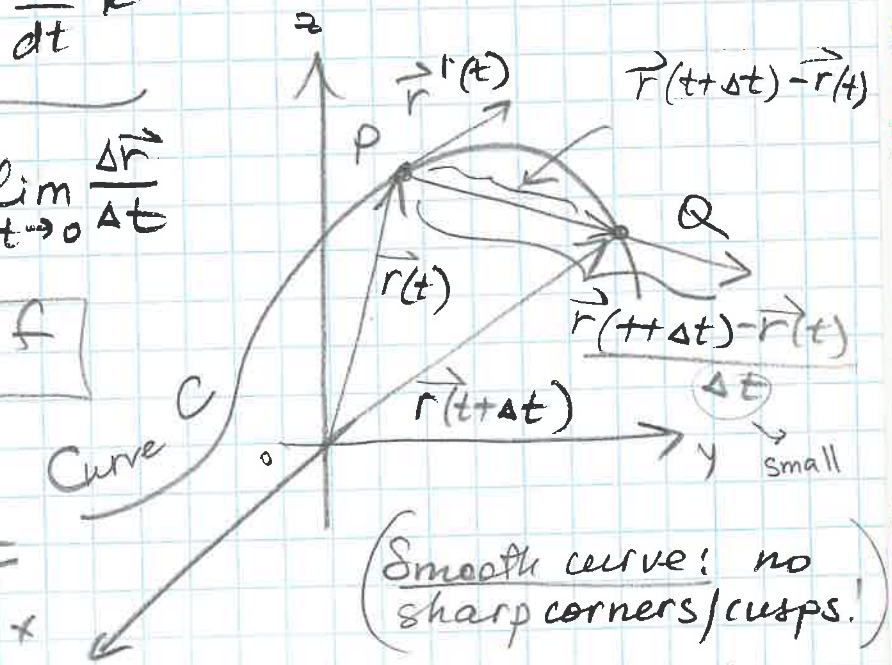
$$= \left[f(t+\Delta t)\vec{i} + g(t+\Delta t)\vec{j} + h(t+\Delta t)\vec{k}\right] - \left[f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}\right] =$$

$$= \left[f(t+\Delta t) - f(t)\right]\vec{i} + \left[g(t+\Delta t) - g(t)\right]\vec{j} + \left[h(t+\Delta t) - h(t)\right]\vec{k}$$

$$\Rightarrow \lim_{\Delta t \to 0} \frac{\vec{\Delta r}}{\Delta t} = \left[\lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}\right]\vec{i} + \left[\lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t}\right]\vec{j} + \left[\lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t}\right]\vec{k}$$

$$= \underbrace{\frac{df}{dt}\vec{i} + \frac{dg}{dt}\vec{j} + \frac{dh}{dt}\vec{k}}_{\vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{\Delta r}}{\Delta t}}$$

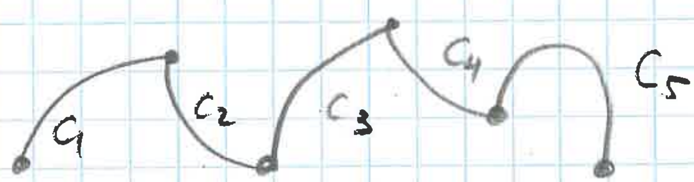
derivative of $\vec{r}(t)$ at t



The curve traced by $\vec{r}$ is <u>smooth</u> if $\frac{d\vec{r}}{dt} \neq \vec{0}$ & continuous (i.e. f, g, h have derivatives that are not 0 at the same time)

(Smooth curve: no sharp corners/cusps.)

- $\vec{r}(t)$ is differentiable if it has derivative at every pt in its domain.
- Nonzero vector $\vec{r}'(t)$ is called the <u>tangent vector</u> to the curve at a pt.
- Tangent line to the curve at a point $(f(t_0), g(t_0), h(t_0))$ is the line through the pt parallel to $\vec{r}'(t_0)$.
- A curve made up of a finite # of smooth curves connected in a continuous fashion is called piecewise smooth:



If $\vec{r}(t)$ is the position vector of a particle moving along a smooth curve in space, then:

$\vec{v}(t) = \frac{d\vec{r}}{dt}$ - velocity vector

$|\vec{v}(t)|$ - speed

$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$ - acceleration

$\frac{\vec{v}}{|\vec{v}|}$ - the direction of motion,

Note:

$$\vec{v} = |\vec{v}| \left(\frac{\vec{v}}{|\vec{v}|} \right)$$

speed direction

<u>Example 3:</u> $\vec{r}(t) = (1+t)\vec{i} + \frac{t^2}{\sqrt{2}}\vec{j} + \frac{t^3}{3}\vec{k}$

a) Speed at $t=1$? $\vec{v}(t) = \vec{i} + \frac{2}{\sqrt{2}}t\vec{j} + t^2\vec{k}$

$\Rightarrow |\vec{v}(t)| = \sqrt{1+2t^2+t^4} \Rightarrow |\vec{v}(1)| = \sqrt{1+2+1} = 2$

(b) $\vec{v}(1) = |\vec{v}(1)| \left(\frac{\vec{v}(1)}{|\vec{v}(1)|} \right) = 2 \cdot \left(\frac{\vec{i}}{2} + \frac{1}{\sqrt{2}}\vec{j} + \frac{1}{2}\vec{k} \right)$

(c) $\vec{a}(t) = 0\vec{i} + \sqrt{2}\vec{j} + 2t\vec{k} \Rightarrow \vec{a}(1) = \sqrt{2}\vec{j} + 2\vec{k}$

- Vector Functions: <u>Differentiation Rules</u>

$\vec{u}(t), \vec{v}(t)$ diff. func's of t

$\vec{c}$ constant vector, k scalar, f any scalar func.

(1) Constant Func Rule: $\frac{d}{dt}(\vec{c}) = \vec{0}$

(2) Scalar Multiple Rules: $\frac{d}{dt}(k\vec{u}(t)) = k\vec{u}\,'(t)$

and $\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$

(3) Sum/Difference Rules: $\frac{d}{dt}(\vec{u}(t) \pm \vec{v}(t)) = \vec{u}\,'(t) \pm \vec{v}\,'(t)$

(4) Dot Product Rule: $\frac{d}{dt}[\vec{u}(t)\cdot\vec{v}(t)] =$

$= \vec{u}\,'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}\,'(t)$

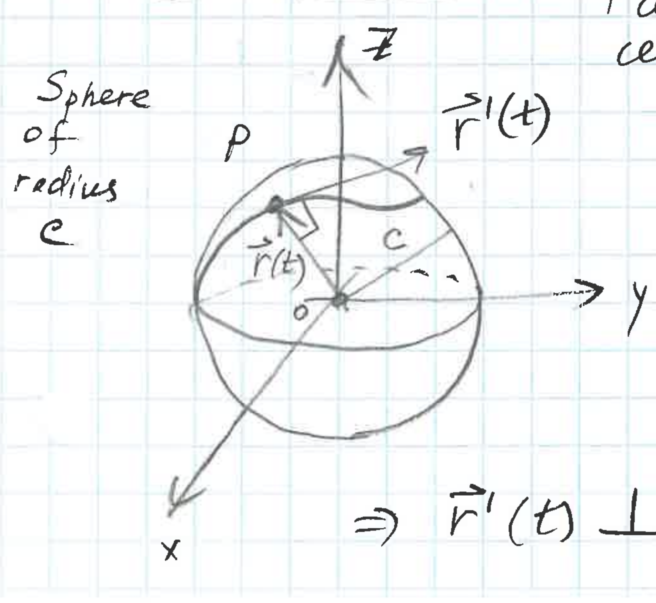
(5) Cross Product Rule: $\frac{d}{dt}[\vec{u}(t)\times\vec{v}(t)] =$

$= \vec{u}\,'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}\,'(t)$ (order matters!)

(6) Chain Rule: $\frac{d}{dt}[\vec{u}(f(t))] = f'(t)\vec{u}\,'(f(t))$

→ Read proofs of (4), (5), (6) on p. 647.

<u>Example 4</u>



Particle is moving on a sphere centered at 0, $|\vec{r}(t)| = c$ (const.) $\sqrt{\vec{r}(t)\cdot\vec{r}(t)}$

$\vec{r}(t)\cdot\vec{r}(t) = |\vec{r}(t)|^2 = c^2$

$\frac{d}{dt}(\vec{r}(t)\cdot\vec{r}(t)) = 0$ (⇒ by (4))

$\vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t) = 0$

$\Rightarrow 2\vec{r}\,'(t)\cdot\vec{r}(t) = 0$

$\Rightarrow \vec{r}\,'(t) \perp \vec{r}(t)$ (Tangent vector (velocity) is orthogonal to position vector of a particle moving on a sphere)

§12.2 Integrals of Vector Functions

Def: The indefinite integral of $\vec{r}$ w.r.t. t is the set of all antiderivatives of $\vec{r}$, denoted by $\int \vec{r}(t)\,dt$. If $\vec{R}(t)$ is any antiderivative of $\vec{r}(t)$ then $\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{C}$
($\vec{R}'(t) = \vec{r}(t)$) const. vector

Def: If the components of $\vec{r}(t)$, $f(t), g(t)$, and $h(t)$ are integrable over $[a,b]$, then so is $\vec{r}$, and

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\vec{i} + \left(\int_a^b g(t)\,dt\right)\vec{j} + \left(\int_a^b h(t)\,dt\right)\vec{k}$$

FTC: (fundamental thm of Calc)

$$\int_a^b \vec{r}(t)\,dt = \vec{R}(t)\Big]_a^b = \vec{R}(b) - \vec{R}(a)$$

Example 1: If $\vec{r}(t) = (2\cos t)\vec{i} + (\sin t)\vec{j} + (2t)\vec{k}$ then

(a) $\int \vec{r}(t)\,dt = \left(\int 2\cos t\,dt\right)\vec{i} + \left(\int \sin t\,dt\right)\vec{j} + \left(\int 2t\,dt\right)\vec{k} =$
$= (2\sin t)\vec{i} - (\cos t)\vec{j} + (t^2)\vec{k} + \vec{C}$ (check by differentiation)

(b) $\int_0^{\pi/2} \vec{r}(t)\,dt = \left[(2\sin t)\vec{i} - (\cos t)\vec{j} + (t^2)\vec{k}\right]_0^{\pi/2}$

$= \left[(2\sin \pi/2)\vec{i} - (\cos \pi/2)\vec{j} + (\pi/2)^2\vec{k}\right] - \left[(2\sin 0)\vec{i} - (\cos 0)\vec{j} + 0\vec{k}\right]$

$= 2\vec{i} + \vec{j} + \dfrac{\pi^2}{4}\vec{k}.$

→ Read: 1) Example 3, p. 651
2) Ideal Projectile Motion, pp. 652-654.