§ 12.3 Arc Length in Space.

(§ 12.3 & § 12.4 are on math features of a curve's shape)

Recall: (Ch. 10) $\mathbb{R}^2$ $x = f(t)$, $y = g(t)$, $a \le t \le b \Rightarrow$ the length of the curve is given by

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt.$$

$\mathbb{R}^3$: consider a smooth curve $\vec{r}(t)$:

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

recall: $\frac{d\vec{r}}{dt}$ is continuous and $\frac{d\vec{r}}{dt} \ne 0$ (no breaks, cusps, corners)

Recall that $\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$

$$|\vec{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Def: The length of a smooth curve $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, $a \le t \le b$, that is traced exactly once as t goes from a to b, is

$$L = \int_a^b |v(t)|\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Example 1 Helix $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + (t)\vec{k}$

Find the length of the helix from the pt. $(1,0,0)$ to pt. $(1,0,2\pi)$

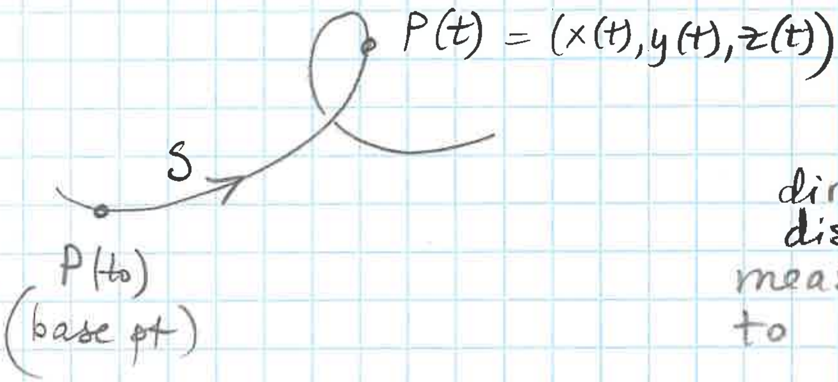
$$L = \int_0^{2\pi} \left|\frac{d\vec{r}}{dt}\right| dt = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{2}\, dt = \boxed{2\sqrt{2}\pi}$$

Note: $(1,0,0) \Rightarrow t = 0$
$(1,0,2\pi) \Rightarrow t = 2\pi$

- Length can be found as a function of t for any pt. on curve starting from a base pt. $P(t_0) = (x(t_0), y(t_0), z(t_0))$ by

$$\boxed{s(t) = \int_{t_0}^{t} |\vec{v}(\tau)|\, d\tau}$$



directed distance: measured from $P(t_0)$ to a pt. $P(t)$

dummy variable

$t > t_0 \Rightarrow s(t) =$ distance along the curve from $P(t_0)$ to $P(t)$

$t < t_0 \Rightarrow s(t) = \ominus$ distance

$s(t)$ values determine pts on the curve $\Rightarrow$ this parameterizes the curve w.r.t. s.

s is called an arc length parameter for the curve.

$s \nearrow$ as $t \nearrow$

If $\vec{r}(t)$ determines a curve, and $s(t)$ is the arc length function of $t \Rightarrow$ one can solve for t as a func. of s: $t = t(s) \Rightarrow$

$\vec{r} = \vec{r}(t(s))$ gives a new parameterization of the curve in terms of s.

Example 2: Take $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + (t)\vec{k}$ (helix)

& $t_0 = 0$. Then $\underset{\text{arc length from } t_0 \text{ to } t}{s(t)} = \int_{t_0}^{t} \underbrace{|\vec{v}(\tau)|}_{\sqrt{2}}\, d\tau$ → see Example 1

$= \int_0^t \sqrt{2}\, d\tau = \sqrt{2}\, t \Rightarrow t = \dfrac{s}{\sqrt{2}} \Rightarrow$

Substituting into $\vec{r}$ gives:

$$\vec{r}(t(s)) = \left(\cos\frac{s}{\sqrt{2}}\right)\vec{i} + \left(\sin\frac{s}{\sqrt{2}}\right)\vec{j} + \left(\frac{s}{\sqrt{2}}\right)\vec{k}.$$

<u>Note:</u> $\frac{ds}{dt} = |\vec{v}(t)|$ (FTC) $\left[\frac{d}{dt}\int_{t_0}^{t}|\vec{v}(\tau)|\,d\tau = |\vec{v}(t)|\right]$

Since $\vec{r}$ is a smooth curve, $\frac{ds}{dt}\left(= |\vec{v}(t)|\right) > 0$ (never zero) $\Rightarrow$ again, $s \nearrow$ (as $t \nearrow$)

<u>Unit Tangent Vector:</u> Curve $\vec{r}(t)$

$\vec{v} = \frac{d\vec{r}}{dt}$ is the tangent vector to the curve.

$\vec{v} = \underbrace{|\vec{v}|}_{\text{speed}}\underbrace{\left(\frac{\vec{v}}{|\vec{v}|}\right)}_{\text{direction of motion}}$. We call $\boxed{\vec{T} = \frac{\vec{v}}{|\vec{v}|}}$ the <u>unit tangent vector</u>

<u>Example 3:</u> $\vec{r}(t) = (1+3\cos t)\vec{i} + (3\sin t)\vec{j} + t^2\vec{k}$

$\Rightarrow \vec{v}(t) = (-3\sin t)\vec{i} + (3\cos t)\vec{j} + (2t)\vec{k}$

$|\vec{v}(t)| = \sqrt{9\sin^2 t + 9\cos^2 t + 4t^2} = \sqrt{9+4t^2} \Rightarrow$

$$\vec{T} = \left(\frac{-3\sin t}{\sqrt{9+4t^2}}\right)\vec{i} + \left(\frac{3\cos t}{\sqrt{9+4t^2}}\right)\vec{j} + \left(\frac{2t}{\sqrt{9+4t^2}}\right)\vec{k}$$

- $\boxed{Q:}$ How does the position vector $\vec{r}(t)$ change w.r.t. arc length s?

That is, what is $\frac{d\vec{r}}{ds}$?

Recall: $\frac{ds}{dt} = |\vec{v}(t)| > 0 \Rightarrow s\nearrow \Rightarrow s$ is one-to-one & invertible

$\Rightarrow t$ is a differentiable func. of s w/ $\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}}$

$= \frac{1}{|\vec{v}(t)|} \Rightarrow \frac{d\vec{r}}{ds} = \underbrace{\frac{d\vec{r}}{dt}\cdot\frac{dt}{ds}}_{\text{chain rule}} = \vec{v}\cdot\frac{1}{|\vec{v}|} = \vec{T}$, i.e., $\boxed{\frac{d\vec{r}}{ds} = \vec{T}}$