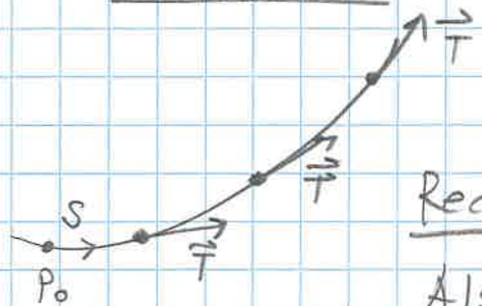
§12.4 Curvature & Normal Vectors of a Curve

Curvature

$\mathbb{R}^2$ or $\mathbb{R}^3$



Smooth curve $\vec{r}(t)$ $\left(\frac{d\vec{r}}{dt} \text{ cont.}, \ \frac{d\vec{r}}{dt} \neq \vec{0} \right)$

Recall: $\vec{T} = \frac{\vec{v}}{|\vec{v}|}$ unit tangent vector

Also: $\vec{T} = \frac{d\vec{r}}{ds}$, where $\underbrace{s(t) = \int_{t_0}^{t} |\vec{v}(\tau)| d\tau}_{\text{arc length}}$

$\vec{T}$ turns as the curve bends.

$|\vec{T}| = 1$ (constant), only the direction changes.

Curvature is the rate at which $\vec{T}$ turns per unit of length along the curve, i.e.

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| \qquad (\kappa \text{ - "kappa"})$$

κ large: the curve turns sharply
κ small: the curve turns slowly

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| = \underbrace{\left| \frac{d\vec{T}}{dt} \frac{dt}{ds} \right|}_{\text{chain rule}} = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right| \qquad \left(\text{since } \frac{ds}{dt} = |\vec{v}| \text{ and } \frac{dt}{ds} = 1/(ds/dt) \right)$$

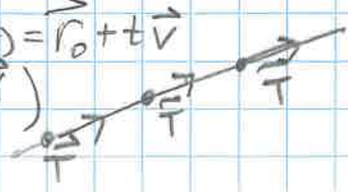
Formula for κ: $\boxed{\kappa = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|}$ (a number)

$\left(\text{where } \vec{T} = \frac{\vec{v}}{|\vec{v}|} \right)$

Example 1 Curvature of a straight line?

$\vec{r}(t) = \vec{r}_0 + t\vec{v}$

$(\vec{r}\,'(t) = \vec{v})$



$\vec{T}$ is a const. vector $\Rightarrow \frac{d\vec{T}}{ds} = \vec{0}$

$\Rightarrow \kappa = |\vec{0}| = 0.$

Example 2. Curvature of a circle of radius a? $(\mathbb{R}^2)$

$$\vec{r}(t) = (a\cos t)\vec{i} + (a\sin t)\vec{j} \Rightarrow \vec{v}(t) = (-a\sin t)\vec{i} + (a\cos t)\vec{j}$$

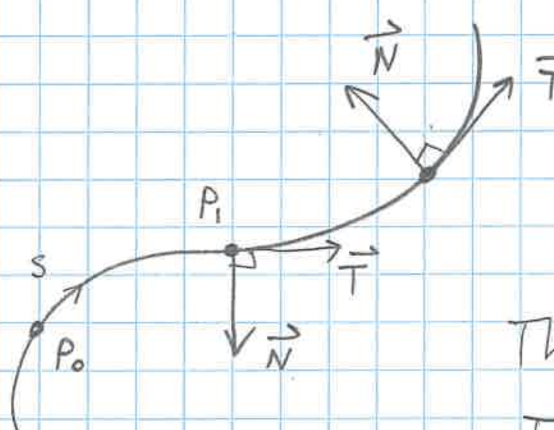
$$\Rightarrow |\vec{v}| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = |a| = a \quad (a>0)$$

$$\Rightarrow T = \frac{\vec{v}}{|\vec{v}|} = (-\sin t)\vec{i} + (\cos t)\vec{j} \Rightarrow \frac{d\vec{T}}{dt} = (-\cos t)\vec{i} - (\sin t)\vec{j}$$

$$\Rightarrow \left|\frac{d\vec{T}}{dt}\right| = \sqrt{(-\cos t)^2 + (-\sin t)^2} = 1 \Rightarrow \kappa = \frac{1}{|\vec{v}|}\frac{d\vec{T}}{dt} = \boxed{\frac{1}{a}}$$

or $\kappa = \dfrac{1}{\text{radius}}$

Normal vector



Recall from §12.1:

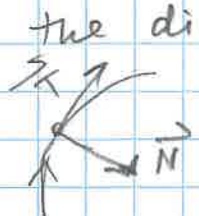
if $\vec{r}(t)$ has $|\vec{r}(t)| = c$ then it can be shown that $\vec{r}(t)\cdot\dfrac{d\vec{r}}{dt} = 0$, i.e. $\vec{r}(t) \perp \dfrac{d\vec{r}}{dt}$

Thus, since $|\vec{T}| = 1 \Rightarrow$

$$T \perp \frac{d\vec{T}}{ds} \quad \left(\vec{T}\cdot\frac{d\vec{T}}{ds} = 0\right)$$

- $\dfrac{d\vec{T}}{ds}$ points in the direction in which the curve is turning.

Recall: $\left|\dfrac{d\vec{T}}{ds}\right| = \kappa$, curvature, so, the unit vector in the direction of $\dfrac{d\vec{T}}{ds}$ is:



$$\boxed{\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds}} \quad (\kappa \neq 0)$$

$\longrightarrow$ principal unit normal vector for a smooth curve.

(points towards the concave side)

Let us find a more useful formula for $\vec{N}$:

$$\vec{N} = \frac{1}{\kappa}\cdot\frac{d\vec{T}}{ds} = \frac{d\vec{T}/ds}{\underbrace{|d\vec{T}/ds|}_{=\kappa}} = \underbrace{\frac{(d\vec{T}/dt)(dt/ds)}{|(d\vec{T}/dt)(dt/ds)|}}_{\text{chain rule}}$$

$$= \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$\frac{dt}{ds} > 0 \rightarrow$ cancels!

So, $\boxed{\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}}$ ($\vec{T}$ is the unit tangent vector)

<u>Example 3:</u> Find $\vec{T}, \vec{N}$, and κ for the curve

$$\vec{r}(t) = (6\sin 2t)\vec{i} + (6\cos 2t)\vec{j} + (5t)\vec{k}$$

<u>Sol.:</u> $\underbrace{\vec{T} = \frac{\vec{v}}{|\vec{v}|}, \quad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}}_{\text{vectors}}, \quad \underbrace{\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|}_{\text{scalar}}$

$$\vec{v} = (12\cos 2t)\vec{i} - (12\sin 2t)\vec{j} + 5\vec{k}$$

$$|\vec{v}| = \sqrt{144\cos^2 2t + 144\sin^2 2t + 25} = \sqrt{169} = 13$$

$$\vec{T} = \left(\frac{12}{13}\cos 2t\right)\vec{i} - \left(\frac{12}{13}\sin 2t\right)\vec{j} + \frac{5}{13}\vec{k}$$

$$\frac{d\vec{T}}{dt} = \left(-\frac{24}{13}\sin 2t\right)\vec{i} - \left(\frac{24}{13}\cos 2t\right)\vec{j} + 0\vec{k}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\left(\frac{24}{13}\right)^2\sin^2 2t + \left(\frac{24}{13}\right)^2\cos^2 2t} = \frac{24}{13}$$

$$\vec{N} = (-\sin 2t)\vec{i} - (\cos 2t)\vec{j}$$

$$\kappa = \frac{1}{13}\cdot\frac{24}{13} = \frac{24}{169}$$

<u>Done!</u>

Note: radius of curvature $\rho = \frac{1}{\kappa}$
(simply the radius of the osculating circle)