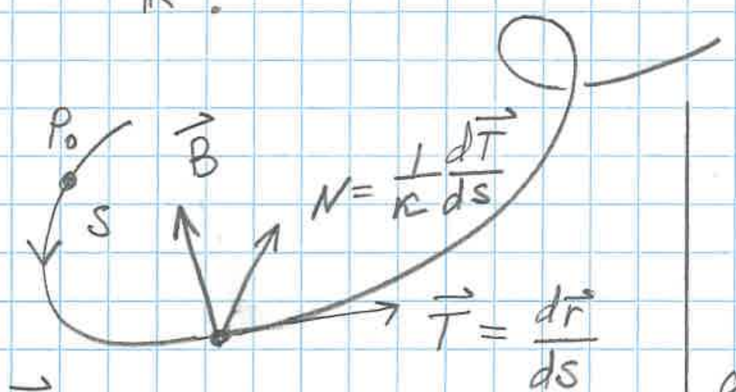
§ 12.5 Tangential & Normal Components of Acceleration.

$\mathbb{R}^3$:



$\vec{T}$: which way you're going
$\vec{N}$: which way you're turning
$\vec{B}$: how curve twists out of $\vec{T}\vec{N}$-plane

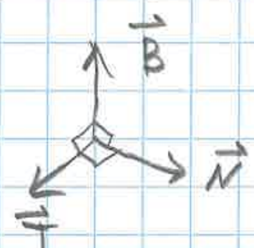
unit tang. vec. principal normal vec, (unit)

$\vec{B} = \vec{T} \times \vec{N}$ is the <u>binormal vector</u> of a curve in $\mathbb{R}^3$.

$\vec{T}, \vec{N}, \vec{B}$ together define a moving right-handed <u>vector frame</u>, the <u>$\vec{T}\vec{N}\vec{B}$-frame</u>.

(or the <u>Frenet frame</u>)

$\vec{T}, \vec{N}, \vec{B}$ are mutually orthogonal unit vectors:



- <u>Tangential & Normal Components of $\vec{a}$</u>:

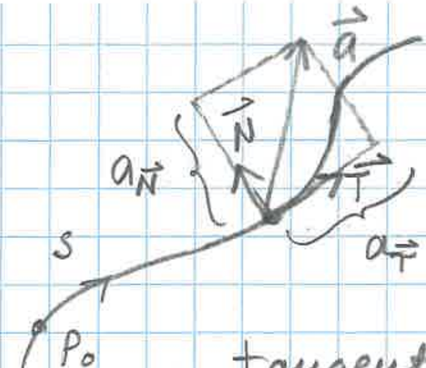
$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt}$$

Want to know how much of $\vec{a}$ acts in the direction of motion, i.e, $\vec{T}$.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}\left(\underbrace{\frac{d\vec{r}}{ds}}_{\vec{T}}\frac{ds}{dt}\right) = \frac{d}{dt}\left(\vec{T}\frac{ds}{dt}\right)$$

$$\underset{\text{product rule}}{=} \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\frac{d\vec{T}}{dt} = \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\left(\frac{d\vec{T}}{ds}\frac{ds}{dt}\right)$$

$$= \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\left(\underbrace{\kappa\vec{N}}_{\text{"}d\vec{T}/ds\text{"}}\frac{ds}{dt}\right) = \boxed{\frac{d^2s}{dt^2}\vec{T} + \kappa\left(\frac{ds}{dt}\right)^2\vec{N}}$$



<u>Def:</u>   scalars

If $\vec{a} = a_T \vec{T} + a_N \vec{N}$, then

tangential component $a_T = \dfrac{d^2 s}{dt^2} = \dfrac{d}{dt} |\vec{v}|$

and normal component $a_N = \kappa \left(\dfrac{ds}{dt}\right)^2 = \kappa |\vec{v}|^2$

(No $\vec{B}$ in the equation: $\vec{a}$ always lies in the $\vec{T}\vec{N}$ plane.)

- a_T measures the rate of change of the length of $\vec{v}$ (change in speed).
- a_N measures the rate of change of the direction of $\vec{v}$.

$\longrightarrow$ $a_N = \kappa |\vec{v}|^2$ is large for sharp turns (large κ) at high speed. (large $|\vec{v}|$)

- Around a circle: if $|\vec{v}| = \text{const} \Rightarrow a_T = 0$, $a_N \neq 0 \Rightarrow \vec{a}$ points along $\vec{N}$ towards the circle's center.
- $a_T \neq 0$ for speeding up or slowing down.

<u>In practice:</u> ~~use~~ find $a_T = \dfrac{d}{dt} |\vec{v}|$ first,

then $a_N = \sqrt{|\vec{a}|^2 - a_T^2}$

(comes from $|\vec{a}|^2 = \vec{a} \cdot \vec{a} = a_T^2 \underbrace{\vec{T} \cdot \vec{T}}_{=1} + a_N^2 \underbrace{\vec{N} \cdot \vec{N}}_{=1} = a_T^2 + a_N^2$)

(no need to compute κ)

Example 1 Without finding $\vec{T}$ and $\vec{N}$, write $\vec{a}$ in the form $a_T\vec{T} + a_N\vec{N}$ for

$$\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} - (t^2)\vec{k}$$

(If you need to find $\vec{T}$ & $\vec{N}$, use $\vec{T} = \frac{\vec{v}}{|\vec{v}|}$ and $\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$)

$$\vec{v} = (-\sin t)\vec{i} + (\cos t)\vec{j} - (2t)\vec{k}$$

$$|\vec{v}| = \sqrt{1+4t^2}$$

$$a_T = \frac{d}{dt}|\vec{v}| = \frac{8t}{2\sqrt{1+4t^2}} = \frac{4t}{\sqrt{1+4t^2}}$$

$$\vec{a} = (-\cos t)\vec{i} - (\sin t)\vec{j} - 2\vec{k}$$

$$|\vec{a}| = \sqrt{1+(-2)^2} = \sqrt{5} \Rightarrow \underset{\sqrt{|\vec{a}|^2 - a_T^2}}{\overset{\parallel}{a_N}} = \sqrt{5 - \frac{16t^2}{1+4t^2}}$$

$$= \sqrt{\frac{5+4t^2}{1+4t^2}}$$

Thus, $\vec{a} = \left(\frac{4t}{\sqrt{1+4t^2}}\right)\vec{T} + \left(\sqrt{\frac{5+4t^2}{1+4t^2}}\right)\vec{N}$

Ex: $t=0 \Rightarrow \vec{a}(0) = 0\cdot\vec{T} + \sqrt{5}\,\vec{N}$

Also: read Example 1, p. 668.