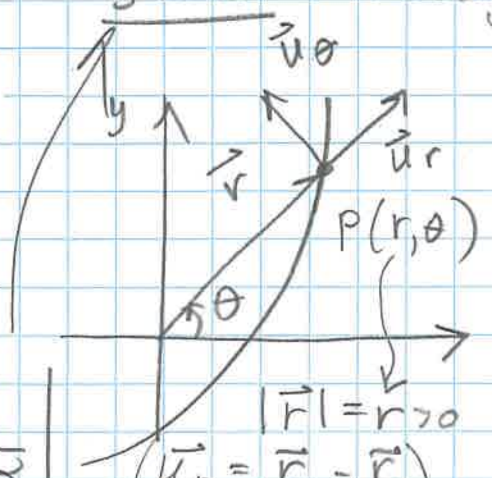
§12.6 Polar Coordinates: $\vec{v}$ and $\vec{a}$

used for calculating paths of planets & satellites in space



$|\vec{r}| = r > 0$

$\left(\vec{u}_r = \dfrac{\vec{r}}{|\vec{r}|} = \dfrac{\vec{r}}{r}\right)$

Optional

Consider unit vectors ($\mathbb{R}^2$)

$\vec{u}_r = (\cos\theta)\vec{i} + (\sin\theta)\vec{j}$ and

$\vec{u}_\theta = (-\sin\theta)\vec{i} + (\cos\theta)\vec{j}$

Note: $\vec{r} = r\vec{u}_r$; $\vec{u}_\theta \perp \vec{u}_r$, $\vec{u}_\theta$ points in the direction of increasing θ.

$$\frac{d\vec{u}_r}{d\theta} = (-\sin\theta)\vec{i} + (\cos\theta)\vec{j} = \vec{u}_\theta$$

$$\frac{d\vec{u}_\theta}{d\theta} = (-\cos\theta)\vec{i} - (\sin\theta)\vec{j} = -\vec{u}_r$$

How do $\vec{u}_r$ & $\vec{u}_\theta$ change with time t?

$$\dot{u}_r = \frac{d\vec{u}_r}{dt} = \frac{d\vec{u}_r}{d\theta}\cdot\underbrace{\frac{d\theta}{dt}}_{\dot{\theta}} = \dot{\theta}\vec{u}_\theta$$

(Newton's dot notation)

$$\dot{u}_\theta = \frac{du_\theta}{dt} = \frac{d\vec{u}_\theta}{d\theta}\cdot\frac{d\theta}{dt} = -\dot{\theta}\vec{u}_r$$

Then $\vec{v} = \dfrac{d\vec{r}}{dt}$ or $= \dot{\vec{r}} = \dfrac{d}{dt}(r\vec{u}_r) = \dot{r}\vec{u}_r + r\dot{\vec{u}}_r$

$$= \dot{r}\vec{u}_r + r\dot{\theta}\vec{u}_\theta$$

and $\vec{a} = \dot{\vec{v}} = (\ddot{r}\vec{u}_r + \dot{r}\dot{\vec{u}}_r) + (\dot{r}\dot{\theta}\vec{u}_\theta + r\ddot{\theta}\vec{u}_\theta + r\dot{\theta}\dot{\vec{u}}_\theta)$

$$= (\ddot{r} - r\dot{\theta}^2)\vec{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{u}_\theta.$$

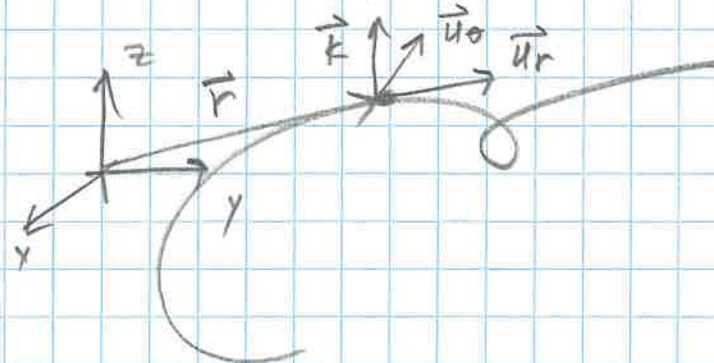
In $\mathbb{R}^3$, $\vec{r} = r\vec{u}_r + z\vec{k}$ (cylindrical coord's)

$\Rightarrow \vec{v} = \dot{r}\vec{u}_r + r\dot{\theta}\vec{u}_\theta + \dot{z}\vec{k}$

$\Rightarrow \vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{u}_\theta + \ddot{z}\vec{k}$

$\vec{u}_r, \vec{u}_\theta, \vec{k}$ form a right-handed frame.

$\vec{u}_r \times \vec{u}_\theta = \vec{k}$, $\vec{u}_\theta \times \vec{k} = \vec{u}_r$, $\vec{k} \times \vec{u}_r = \vec{u}_\theta$



Optional reading: pp. 670 – 672

Three Kepler's Laws (lots of fun!)