Chapter 13 Partial Derivatives

§13.1 Functions of Several Variables

Past: $y = x^2$, $A = \pi r^2$ (circle), $V = x^3$ (cube), etc

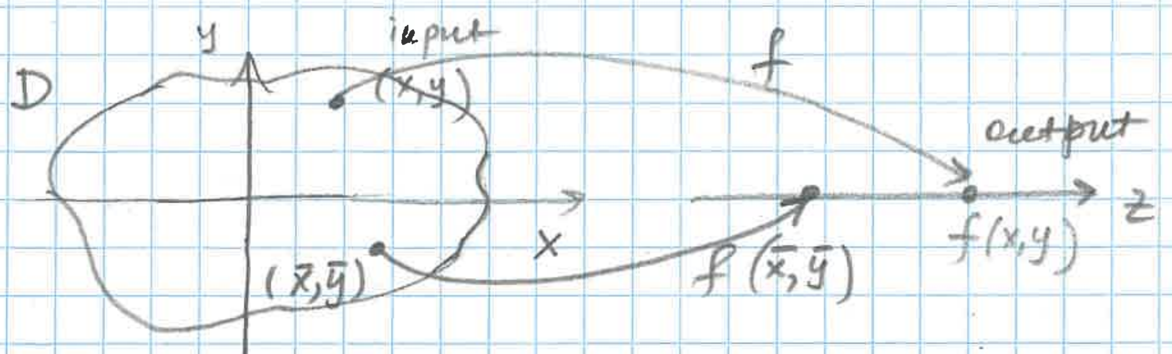
How about $V(r,h) = \pi r^2 h$? (right circular cylinder) $\rightsquigarrow$ depend on both r & h.

Def: Let D be a set of n-tuples of real numbers $(x_1, x_2, \dots, x_n)$. A real-valued func. f on D is a rule that assigns a unique real number $w = f(x_1, \dots, x_n)$ to each element in D.

$$(x_1, x_2, \dots, x_n) \xrightarrow{\text{input}} \boxed{f} \xrightarrow{\text{output}} w$$

D is the func.'s domain. The set of all w's is func.'s range. The symbol w is the dependent variable of f, which is a func. of n independent variables $x_1, x_2, \dots, x_n$.

Ex: $z = f(x,y)$ – a func. of 2 variables
z is dependent, x, y are indep. var's.



Ex: $V(r,h) = \pi r^2 h$ $\qquad$ input $(r,h) \to V \to$ output $V(r,h)$ $\qquad r, h > 0$

Ex: $f(x,y) = \sqrt{y-x}$ $\qquad$ Domain: $y - x \geq 0$ or $y \geq x$

Range: $z = \sqrt{y-x} \geq 0$

$f(3,12) = \sqrt{12-3} = \sqrt{9} = 3$

Ex: $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2}$

Domain: $x^2+y^2+z^2 \neq 0$, i.e. $(x,y,z) \neq (0,0,0)$

Range: $(0,\infty)$

Ex: $w = xy \ln z$

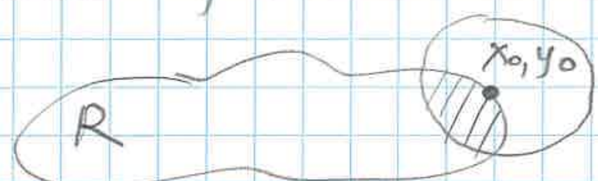
Domain: $z > 0$
Range: $(-\infty,\infty)$

- Functions of Two Variables:

Def: • A pt. (x_0,y_0) in a region R in the xy-plane is an interior pt. of R if it is a center of a disk that lies entirely in R.

• The interior of R is the set of all interior pts of R

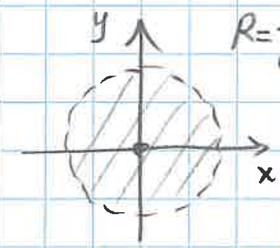
• A pt. (x_0,y_0) is a boundary pt. of R if every disk centered at (x_0,y_0) contains points both points from inside and outside of R.



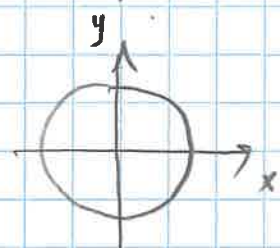
• All boundary pts makeup the boundary of R

- The region is closed if it contains all its boundary pts.
- The region is open if it consists entirely of interior pts.

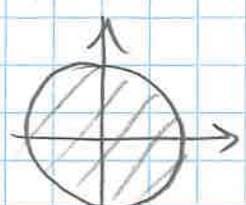
Examples:



$R = \{(x,y) : x^2+y^2 < 1\}$ – open unit disk (every pt. is an interior pt.)



$R = \{(x,y) : x^2+y^2 = 1\}$ – boundary of unit disk (the unit circle)

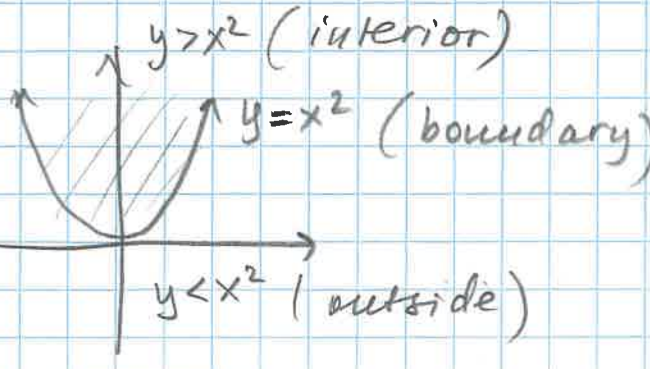


$R = \{(x,y) : x^2+y^2 \leq 1\}$ – closed unit disk

Def: • A region in the plane is bounded if it lies inside a disk of fixed radius. (ex: line segments, Δ's, ellipses, disks, ...)

• A region is unbounded if it is not bounded (ex: lines, quadrants, axes, the xy-plane itself.)

Ex:



$z = \sqrt{y - x^2}$

Domain: $y \geq x^2$

closed, unbounded region

Def: The set of pts in the xy plane where $f(x,y)=c$ (constant) is called a <u>level curve</u> of f.

The set of all pts $(x,y,f(x,y))$ in space, for (x,y) in the domain of f, is called the <u>graph</u> of f, or the <u>surface</u> $z=f(x,y)$.

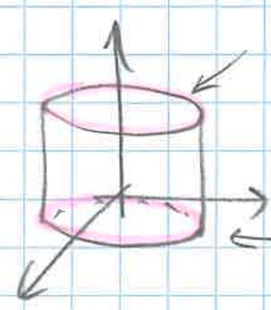
Example: Graph $f(x,y)=100-x^2-y^2$ & level curves $f(x,y)=0$, $f(x,y)=51$ & $f(x,y)=75$

domain D: entire xy-plane, $z=100-x^2-y^2$ is the paraboloid (see Demo-Section13_1.nb)

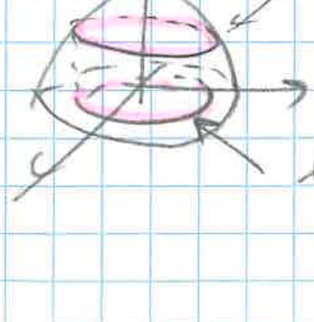
Note: level curves $f(x,y)=c$ lie in xy-plane;

e.g., $100-x^2-y^2=0$ or $100-x^2-y^2=51$

The <u>curve in space</u> in which the plane $z=c$ cuts a surface $z=f(x,y)$ is called the <u>contour curve</u>



$f(x,y)=100-x^2-y^2$



• Functions of Three Variables: $f(x,y,z)$.

Def: The set of pts (x,y,z) in space where a function of (x,y,z) has a constant value $f(x,y,z) = C$ is called a level surface of f.

Note: $f(x,y) = C$ makes a curve in xy-plane

$f(x,y,z) = C$ makes a surface in (3D) domain of f.

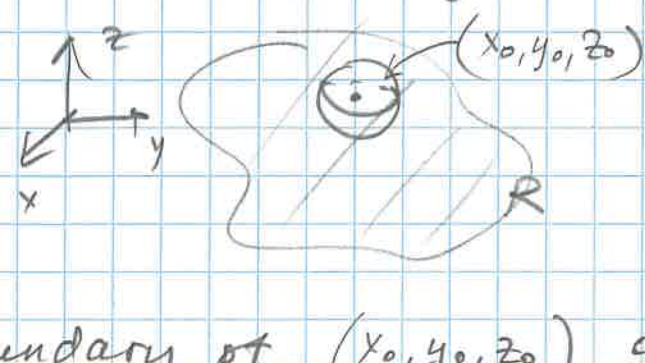
(Graphs of $(x,y,z,f(x,y,z))$ lie in 4D space → can't plot!)

Ex: $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ → describe.

each level surface $\sqrt{x^2+y^2+z^2} = C$

$(c>0)$ is a sphere of radius C, centered at the origin. ($\sqrt{x^2+y^2+z^2} = 0$ gives the origin.)

• Interior pt (x_0,y_0,z_0) in a (3D) region R is a pt which is the center of solid ball that lies entirely in R



• Boundary pt (x_0,y_0,z_0) of R: every solid ball centered at (x_0,y_0,z_0) has both pts from the inside & outside of R

• Region can be closed or open (see p. 650)