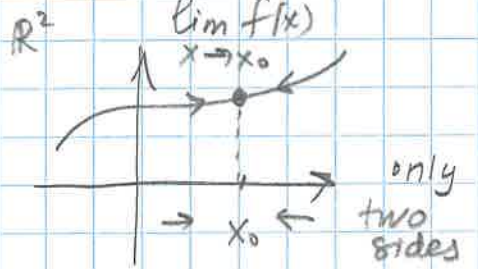
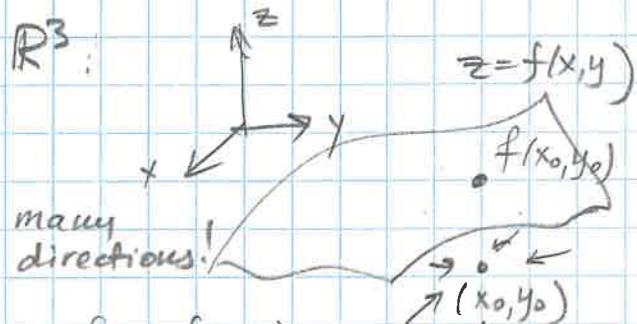
§ 13.2 Limits & Continuity (in Higher Dimensions)

__Idea of limit:__ similar to what we know from Calc I, but now we have more directions to consider!



VS.



for $z=f(x,y)$

$\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$?

__Def:__ We say that a func. $f(x,y)$ approaches the limit L as (x,y) approaches (x_0,y_0) & write

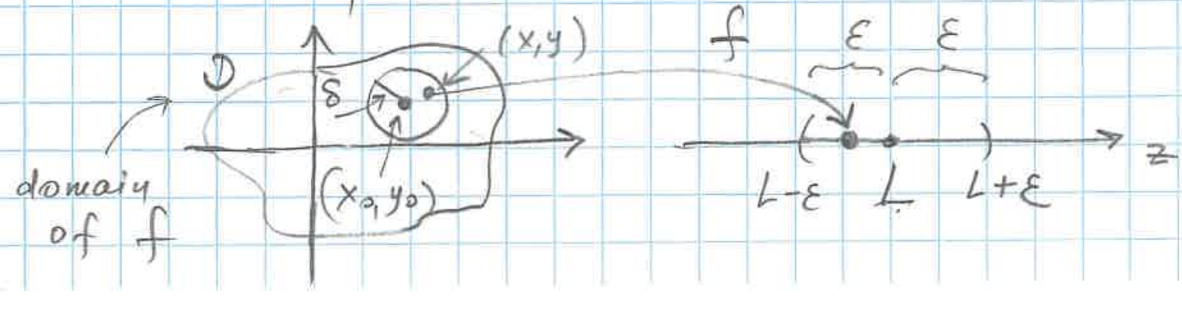
$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{if, for every } \varepsilon>0,$$

there exists $\delta>0$ s.t. for all (x,y) in the domain of f,

$$|f(x,y)-L|<\varepsilon \quad \text{whenever} \quad 0<\underbrace{\sqrt{(x-x_0)^2+(y-y_0)^2}}_{\text{dist. between } (x_0,y_0)\ \&\ (x,y)}<\delta$$

gets arbitrarily small whenever the distance is sufficiently small (but $\neq 0$)

((x_0,y_0) can be either interior or boundary pt. of the domain of f.)



<u>Note</u>: $\underset{(x,y)\to(x_0,y_0)}{\lim} x = x_0$ (plane $z=x$), $\underset{(x,y)\to(x_0,y_0)}{\lim} y = y_0$ (plane $z=y$), $\underset{(x,y)\to(x_0,y_0)}{\lim} k = k$ (plane $z=k$)

$(k \in \mathbb{R})$

- <u>Properties of Limits of $z=f(x,y)$:</u>

Let $\underset{(x,y)\to(x_0,y_0)}{\lim} f(x,y) = L$ & $\underset{(x,y)\to(x_0,y_0)}{\lim} g(x,y) = M$

① Sum / Difference Rules:

$$\lim_{(x,y)\to(x_0,y_0)} \left[f(x,y) \pm g(x,y)\right] = L \pm M$$

② Constant Multiple Rule:

$$\lim_{(x,y)\to(x_0,y_0)} \left[k f(x,y)\right] = kL \quad \text{for any } k \in \mathbb{R}.$$

③ Product Rule:

$$\lim_{(x,y)\to(x_0,y_0)} \left[f(x,y) \cdot g(x,y)\right] = L \cdot M$$

④ Quotient Rule:

$$\lim_{(x,y)\to(x_0,y_0)} \left[f(x,y)/g(x,y)\right] = L/M \quad (M \neq 0)$$

⑤ Power Rule:

$$\lim_{(x,y)\to(x_0,y_0)} \left[f(x,y)\right]^n = L^n, \quad n \text{ is a pos. integer.}$$

⑥ Root Rule:

$$\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}, \quad n \text{ is a pos. integer.}$$

If n even, $L > 0$.

Examples: Evaluate the following limits:

1) $\displaystyle\lim_{(x,y)\to(3,-5)} \frac{x^2-y^2}{x^2+y^2} = \frac{3^2-(-5)^2}{3^2+(-5)^2} = \boxed{-\frac{16}{34}}$

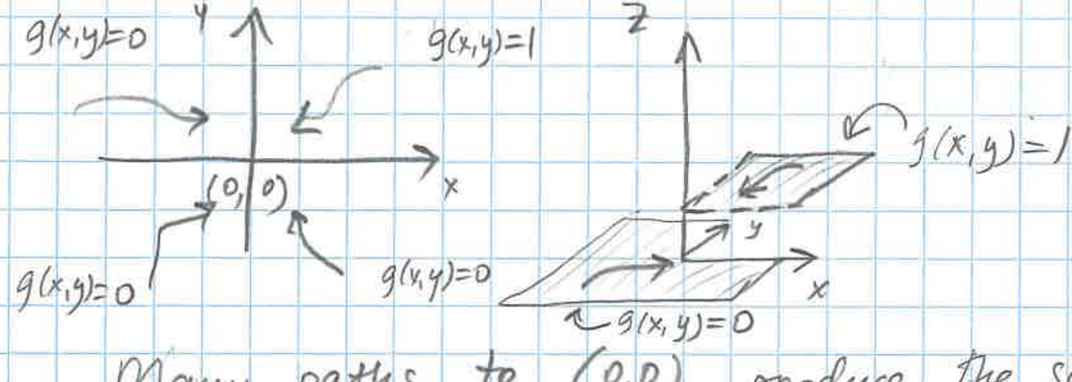
2) $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x-xy+4}{7x^2y+5x-y^3} = \frac{0-0+4}{0+0-1} = \boxed{-4}$

3) $\displaystyle\lim_{(x,y)\to(0,0)} \underbrace{\frac{x^2-xy}{\sqrt{x}-\sqrt{y}}}_{0/0} \overset{\text{conjugate trick}}{=} \lim_{(x,y)\to(0,0)} \underbrace{\frac{(x^2-xy)(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}}_{\text{eliminating 0 in the denominator}} =$

$\displaystyle= \lim_{(x,y)\to(0,0)} \frac{x\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{x-y}} = \lim_{(x,y)\to(0,0)} x(\sqrt{x}+\sqrt{y}) = \boxed{0}$

Note: factor $x-y$ can be canceled since the path $y=x$ (or $x-y=0$) is not in the domain of func. $f(x,y) = \dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}}$!

4) "Pizza slice function": $g(x,y) = \begin{cases} 1, & \text{if } \underbrace{x>0, y>0}_{1^{st}\text{ quad.}} \\ 0, & \text{else} \end{cases}$



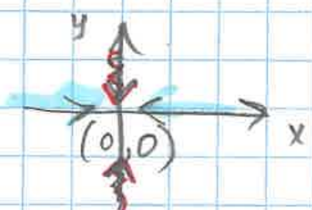
Many paths to (0,0) produce the same limit 0, as $g(x,y)=0$
but if we approach (0,0) from 1^{st} quadrant, we have limit 1, as $g(x,y)=1$ there.

Thus, $\displaystyle\lim_{(x,y)\to(0,0)} g(x,y)$ DNE!

• Two-Path Test for Non existence of a Limit

$\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ DNE if $f(x,y)$ has 2 different limits along 2 different paths in its domain

Example: Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ $= f(x,y)$ DNE.



Domain: $(x,y) \neq (0,0)$

Path 1: approach along y-axis $(x=0)$

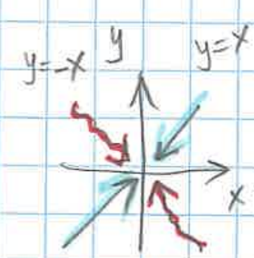
$$\lim_{\substack{(x,y)\to(0,0)\\ x=0\\ (\text{along } y\text{-axis})}} f(x,y) = \lim_{(x,y)\to(0,0)} f(0,y) = \lim_{(x,y)\to(0,0)} \frac{-y^2}{y^2} = \lim_{(x,y)\to(0,0)} (-1) = \boxed{-1}$$

Path 2: approach along x-axis $(y=0)$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0\\ (\text{along } x\text{-axis})}} f(x,y) = \lim_{(x,y)\to(0,0)} f(x,0) = \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2} = \lim_{(x,y)\to(0,0)} 1 = \boxed{1}$$

$$-1 \neq 1 \Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \text{ DNE.}$$

• Some other paths to choose: $y = mx$, $m \in \mathbb{R}$ — i.e., $y=x$ & $y=-x$



Another example: $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ DNE since

Path 1 along $y=x$ $\rightarrow$ $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{2x^2} = \boxed{\dfrac{1}{2}}$, but

Path 2 along $y=-x$ $\rightarrow$ $\lim\limits_{(x,y)\to(0,0)} \dfrac{-x^2}{2x^2} = \boxed{-\dfrac{1}{2}}$

Continuity:

- A function $z = f(x,y)$ is continuous at (x_0, y_0) if $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$ (no "surprises")

- $f(x,y)$ is continuous if it is continuous at every pt. of its domain.

- Composites: if f is cont. at (x_0,y_0) & g is a single-variable func. continuous at $f(x_0,y_0)$, then the composite $(g\circ f)(x) = g(f(x,y))$ is continuous at (x_0,y_0).

  E.g., e^{x-y}, $\cos\frac{xy}{x^2+1}$, $\ln(1+x^2y^2)$ are continuous for any x & y.