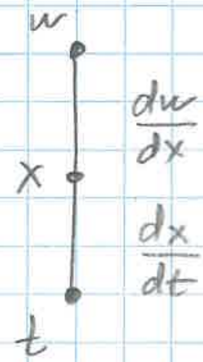
§ 13.4 The Chain Rule.

Calc I: if $w(t) = f(\underbrace{g(t)}_{x})$ then

$w'(t) = f'(\underbrace{g(t)}_{x}) \cdot g'(t)$ or $\dfrac{dw}{dt} = \dfrac{dw}{dx}\dfrac{dx}{dt}$



① Consider $w = f(x, y)$, a func. of 2 var's.

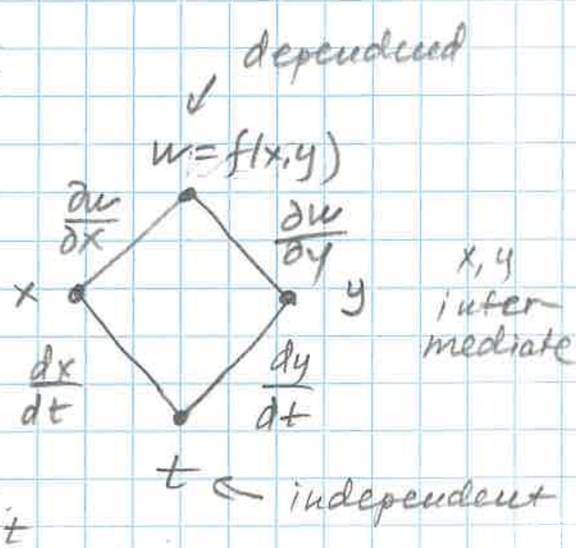
Theorem (5): If $w = f(x,y)$ has continuous f_x and f_y, and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite $w(t) = f(x(t), y(t))$ is a differentiable func. of t and

$$\frac{dw}{dt}\left(= \frac{df}{dt}\right) = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t))\, y'(t)$$

or $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\cdot\dfrac{dy}{dt}$

(Proof is on p. 705)

Branch / Tree Diagram:



$$\underbrace{\frac{dw}{dt}}_{\text{w as a func. of t}} = \underbrace{\frac{\partial w}{\partial x}}_{\text{w as a func. of x \& y}} \cdot \underbrace{\frac{dx}{dt}}_{\text{func. of t}} + \underbrace{\frac{\partial w}{\partial y}}_{w = w(x,y)} \cdot \underbrace{\frac{dy}{dt}}_{\text{func. of t}}$$

Or, even more precise:

$$\frac{dw}{dt}(t_0) = \underset{\left(\text{or } \frac{\partial f}{\partial x}\right)}{\frac{\partial w}{\partial x}}(x_0, y_0)\,\frac{dx}{dt}(t_0) + \underset{\left(\text{or } \frac{\partial f}{\partial y}\right)}{\frac{\partial w}{\partial y}}(x_0, y_0)\,\frac{dy}{dt}(t_0)$$

$$\begin{pmatrix} x_0 = x(t_0) \\ y_0 = y(t_0) \end{pmatrix}$$

Example 1: If $w = x^2 y + 3xy^4$ where $\underbrace{x = \sin 2t,\ y = \cos 2t}_{\text{path in } xy\text{-plane}}$

find $\frac{dw}{dt}$ when $t = 0$.

Solution:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \quad \leftarrow \text{Chain rule}$$

$$= \frac{\partial}{\partial x}(x^2 y + 3xy^4)\cdot\frac{d}{dt}(\sin 2t) + \frac{\partial}{\partial y}(x^2 y + 3xy^4)\frac{d}{dt}(\cos 2t)$$

$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-2\sin 2t)$$

Then $$\left.\frac{dw}{dt}\right|_{t=0} = (2\cdot 0\cdot 1 + 3\cdot 1^4)(2\cos 0) + (0^2 + 12\cdot 0\cdot 1^3)(-2\sin 0)$$

$$= \boxed{6}$$

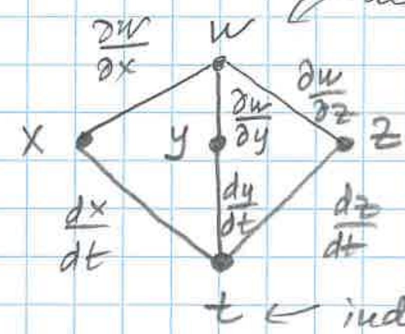
$$\begin{pmatrix} x(0) = \sin 0 = 0 \\ y(0) = \cos 0 = 1 \end{pmatrix}$$

(2) Consider $w = f(x,y,z)$, a func. of 3 var's.

Theorem (6): If $w = f(x,y,z)$ is differentiable and x, y, z are differentiable func's of t, then w is a differentiable func. of t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Branch Diagram:



← dependent

x, y, z - intermediate

← independent

Example 2: Find $\frac{dw}{dt}$ if

(a) $w = z - \sin xy$, where $x = t$, $y = \ln t$, $z = e^{t-1}$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= (-y\cos xy)(1) + (-x\cos xy)\frac{1}{t} + (1)(e^{t-1})$$

OR

(b) Substitute x, y, z into w and find $\frac{dw}{dt}$:

$$w(t) = e^{t-1} - \sin(t\ln t)$$

$$\frac{dw}{dt} = e^{t-1} - \cos(t\ln t)\left[1\cdot\ln t + t\cdot\frac{1}{t}\right]$$

$$= -\underbrace{\ln t}_{y}\cos(\underbrace{t\ln t}_{xy}) - \underbrace{t}_{x}\cos(\underbrace{t\ln t}_{xy})\frac{1}{t} + e^{t-1}$$

Compare (a) & (b) $\rightarrow$ same!

- ## Functions Defined on Surfaces:

Example: if interested in the temperature $w = f(x, y, z)$ at pts (x, y, z) on the earth's surface, then we may prefer to think about x, y, z as functions of longitude r & latitude s:

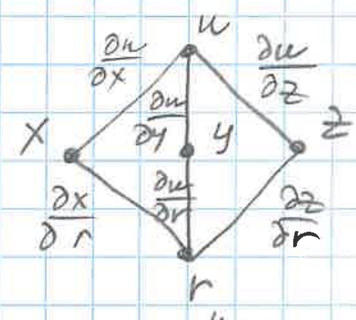
namely, $x = g(r,s)$, $y = h(r,s)$, $z = k(r,s)$

and $\underbrace{w = f(g(r,s), h(r,s), k(r,s))}_{\text{"}}$. <u>Next:</u>

<u>Theorem (7)</u> 2 independent variables & $\to r, s$
3 intermediate variables $\to x, y, z$

Suppose $w = f(x,y,z)$, $x = g(r,s)$, $y = h(r,s)$, and $z = k(r,s)$. If w, x, y, z are differentiable, then w has two partial derivatives:

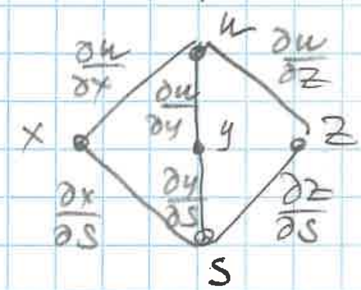
$$\frac{\partial w}{\partial r} = \underbrace{\frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}}$$

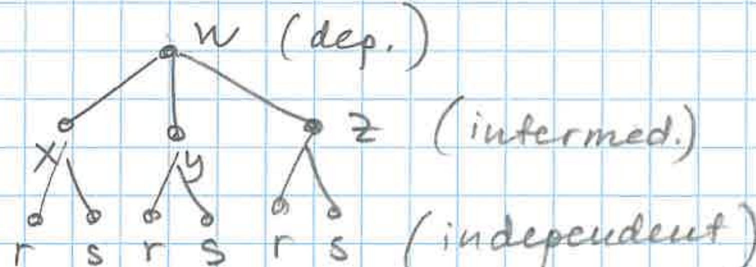


all partial der's!

and

$$\frac{\partial w}{\partial s} = \overbrace{\frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}}$$





<u>Example 3:</u> Express $\frac{\partial w}{\partial r}$ & $\frac{\partial w}{\partial s}$ as func's of r & s by using the Chain rule, if

$w = \ln(x^2+y^2+z^2)$, $x = re^s \sin r$, $y = re^s \cos r$, $z = re^s$

<u>Solution:</u> $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$

$= \frac{2x}{x^2+y^2+z^2} \cdot (e^s \sin r + re^s \cos r) + \frac{2y}{x^2+y^2+z^2}(e^s \cos r -$

$$-re^s \sin r) + \frac{2z}{x^2+y^2+z^2} e^s$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} =$$

$$= \frac{2x}{x^2+y^2+z^2}\left(re^s \sin r\right) + \frac{2y}{x^2+y^2+z^2}\left(re^s \cos r\right) +$$

$$+ \frac{2z}{x^2+y^2+z^2}\left(re^s\right)$$

- General Chain Rule:

Let $w = f(x_1, x_2, \dots, x_n)$ & each x_i, $i=1,\dots,n$, be a function of $t_1, \dots, t_m$. Then

$$w(t_1, \dots, t_m) = f(x_1(t_1, \dots, t_m), \dots, x_n(t_1, \dots, t_m))$$

and $$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \dots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_j},$$

$j = 1, \dots, m$

- Implicit Differentiation (Calc 1 → revisited):

Consider $\underbrace{F(x,y)}_{\text{differentiable}} = 0$ which defines y implicitly as a func. of x, say $y = f(x)$,

i.e. $F(x, f(x)) = 0$, x in the domain of f.

Differentiate w.r.t. x using the chain rule:

$$\frac{\partial F}{\partial x} \cdot \underbrace{\frac{dx}{dx}}_{1} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial y} = -F_x/F_y$$

$(y = f(x))$ $(\text{if } F_y \neq 0)$

Example 4: Find $\frac{dy}{dx}$ if $x^3+y^3=6xy$.

Old way (Calc I)

$$3x^2+3y^2\frac{dy}{dx}=6y+6x\frac{dy}{dx}$$

$$\frac{dy}{dx}(3y^2-6x)=6y-3x^2$$

$$\frac{dy}{dx}=\frac{6y-3x^2}{3y^2-6x}=\frac{2y-x^2}{y^2-2x}$$

New way $\left(\frac{dy}{dx}=-F_x/F_y\right)$

$$F(x,y)=x^3+y^3-6xy=0$$

$$\frac{dy}{dx}=-\frac{3x^2-6y}{3y^2-6x}=\frac{2y-x^2}{y^2-2x}$$

Same!

- More variables? Suppose z is given implicitly as a func. $z=f(x,y)$ by eqn. $F(x,y,z)=0$.

So, $F(x,y,f(x,y))=0$. Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$

Diff. w.r.t. x: (y is a const.)

$$\frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{=1}+\frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{=0}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0$$

$$\Rightarrow \frac{\partial z}{\partial x}=-\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial z}=-F_x/F_z \quad (F_z\neq 0)$$

Similarly, $\frac{\partial z}{\partial y}=-F_y/F_z$

Example 5: Find $\frac{\partial z}{\partial x}$ if $\underbrace{xz+y\ln z-z^2+4}_{F(x,y,z)}=0$ defines $z=z(x,y)$.

$$\frac{\partial z}{\partial x}=-F_x/F_z=\frac{-z}{x+\frac{y}{z}-2z}=\frac{-z^2}{zx+y-2z^2}$$