§13.5 Directional Derivatives and Gradient Vectors.

So far: took a look at f_x & f_y for a func. $z = f(x,y)$, both represent the rate of change of f as we vary x (holding y fixed) and as we vary y (holding x fixed), respectively. What happens if we allow both x & y change simultaneously?

The issue is that there are many ways to do this. Let us define change in f for a chosen (given) direction.

- Directional Derivatives in the Plane:

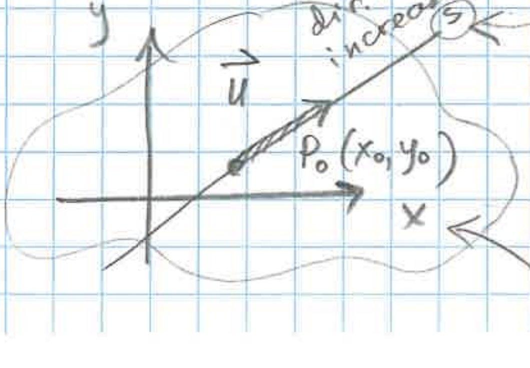
Given curve $\underbrace{x = g(t),\ y = h(t)}_{\text{represents a path w/ parameter } t}$, for a diff. func. $f(x,y)$,

$$\underbrace{\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}_{\text{rate of change of } f \text{ w.r.t. } t} \quad \text{(Chain rule)}$$

$\Rightarrow \frac{df}{dt}$ depends on the direction of motion along $x = g(t)$, $y = h(t)$ (among other things)

- Suppose our curve (path) is a straight line.

$$\left.\begin{aligned} x &= x_0 + su_1 \\ y &= y_0 + su_2 \end{aligned}\right\} \text{ line } \parallel \overbrace{\vec{u} = \langle u_1, u_2 \rangle}^{\text{unit vec!}}, \text{ going through } P_0(x_0, y_0)$$



Let parameter s measure arc length in the direction of $\vec{u}$ from P_0

$\Rightarrow \dfrac{df}{ds}$ is the rate of change of f at P_0 in the direction of $\vec{u}$

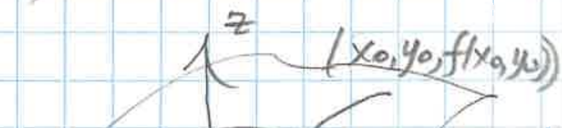
<u>Def:</u> The derivative of f at $P_0(x_0, y_0)$ in the direction of a unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is

$$\left(\frac{df}{ds}\right)_{\vec{u},P_0} = \underbrace{\left(D_{\vec{u}}f\right)_{P_0}}_{\text{notation}} = \lim_{s\to 0} \frac{f(x_0+su_1, y_0+su_2) - f(x_0,y_0)}{s}$$

(provided the limit exists)

Called: the directional derivative.

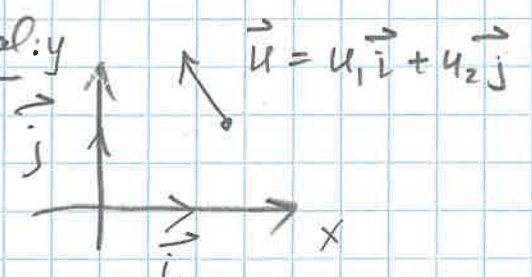
<u>Note:</u> $\dfrac{\partial f}{\partial x}(x_0,y_0)$ & $\dfrac{\partial f}{\partial y}(x_0,y_0)$ are the directional derivatives of f at P_0 at the $\vec{i}$ & $\vec{j}$ directions.

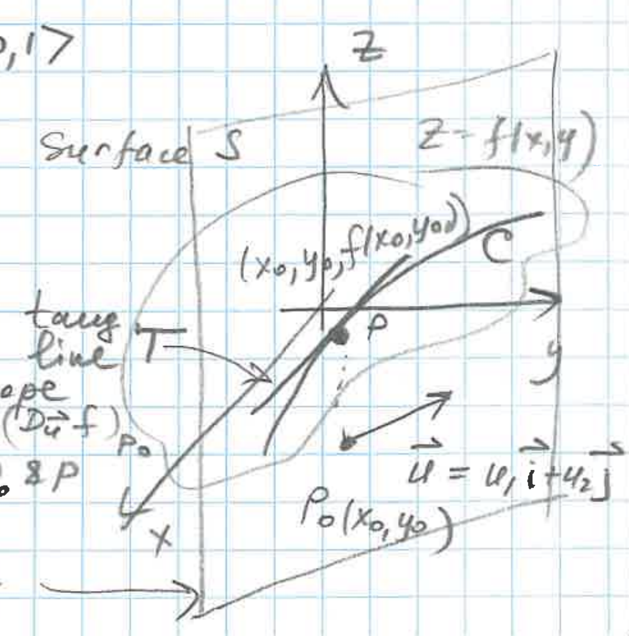


$D_{\vec{i}}f = f_x$, $D_{\vec{j}}f = f_y$

$\vec{u} = \vec{i} = \langle 1, 0\rangle$ $\qquad$ $\vec{u} = \vec{j} = \langle 0, 1\rangle$

<u>In general:</u>





Vertical plane through P_0 & P parallel to $\vec{u}$ intersects surface S in a curve C

Formula for $(D_{\vec{u}} f)_{P_0}$?

$$\left(\frac{df}{ds}\right)_{\vec{u}, P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0} \frac{dx}{ds} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \frac{dy}{ds} \leftarrow \text{chain rule}$$

$$= \underbrace{\left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} u_2}_{\text{dot product of 2 vectors}} \leftarrow \text{comes from } \begin{cases} x = x_0 + s u_1 \\ y = y_0 + s u_2 \end{cases}$$

$$= \underbrace{\left[\left(\frac{\partial f}{\partial x}\right)_{P_0} \vec{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \vec{j}\right]}_{\text{the gradient (vector) of } f \text{ at } P_0} \cdot \underbrace{\left[u_1 \vec{i} + u_2 \vec{j}\right]}_{\text{direction } \vec{u}}$$

Def.: The gradient vector (or, simply, gradient) of $f(x,y)$ at a pt. $P_0(x_0, y_0)$ is

$$\underbrace{\nabla f}_{\text{"grad } f\text{" or "del } f\text{"}} = \left(\frac{\partial f}{\partial x}\right)_{P_0} \vec{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \vec{j}$$

Thus, $(D_{\vec{u}} f)_{P_0} = (\nabla f)_{P_0} \cdot \vec{u}$ (or $D_{\vec{u}} f = \nabla f \cdot \vec{u}$ in brief)

Example 1. Find the derivative of $f(x,y) = x^2 y^3 - 4y$ in the direction of $\vec{v} = 2\vec{i} + 5\vec{j}$, at the pt. $(2,-1)$

Note: $|\vec{v}| \neq 1$, so find $\vec{u} = \frac{\vec{v}}{|\vec{v}|} =$

$$= \frac{\vec{v}}{\sqrt{4+25}} = \frac{\vec{v}}{\sqrt{29}} = \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle$$ (unit vector in the dir. of $\vec{v}$)

$$\nabla f(x,y) = \langle 2xy^3, 3x^2y^2 - 4 \rangle$$

$$\nabla f(2,-1) = \langle 2 \cdot 2 \cdot (-1)^3, 3 \cdot 2^2 \cdot (-1)^2 - 4 \rangle = \langle -4, 8 \rangle$$

So, $D_{\vec{u}} f(2,-1) = \langle -4, 8 \rangle \cdot \langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \rangle = \underbrace{\frac{32}{\sqrt{29}}}_{\text{rate of change of } f \text{ in the dir. of } \vec{u} = \vec{v}/|\vec{v}|}$

→ Note: $D_{\vec{u}} f$ can be found for 3 var's (or more):

$$D_{\vec{u}} f = \langle f_x, f_y, f_z \rangle \cdot \vec{u}$$

Properties of $D_{\vec{u}} f$:

$$D_{\vec{u}} f \overset{def}{=} \nabla f \cdot \vec{u} = |\nabla f| \overbrace{|\vec{u}|}^{1} \cos\theta = |\nabla f| \cos\theta$$

$-1 \le \cos\theta \le 1 \Rightarrow \overbrace{-|\nabla f|}^{min} \le D_{\vec{u}} f \le \overbrace{|\nabla f|}^{max}$ (θ: angle between $\vec{u}$ and ∇f)

① $\theta = 0$ when $\vec{u}$ is in the direction of ∇f $\Rightarrow$ $\cos 0 = 1 \Rightarrow D_{\vec{u}} f = \overset{(max)}{|\nabla f|}$. This means that f increases most rapidly in the direction of ∇f.

② $\theta = \pi$ when $\vec{u}$ is in the direction of $-\nabla f$ $\Rightarrow \cos\pi = -1 \Rightarrow D_{\vec{u}} f = \overbrace{-|\nabla f|}^{min}$. This means that f decreases most rapidly in the dir. of $-\nabla f$.

③ if $\vec{u} \perp \nabla f$ $(\theta = \pi/2 \Rightarrow \cos\frac{\pi}{2} = 0)$ $\Rightarrow D_{\vec{u}} f = 0 \Rightarrow$ no change if in the dir. orthogonal to ∇f.

Example 2: $f(x,y) = x^2 + xy + y^2$, $P_0 = (-1,1)$

$$\nabla f = \langle 2x+y, x+2y \rangle \Rightarrow \nabla f(-1,1) = \langle -1, 1 \rangle$$

Direction of ∇f is $\vec{u} = \frac{\nabla f}{|\nabla f|} = \frac{\langle -1, 1 \rangle}{\sqrt{2}} = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$D_{\vec u} f$ increases most rapidly in the dir. of $\vec u = \langle -\frac{1}{\sqrt 2}, \frac{1}{\sqrt 2} \rangle$, $D_{\vec u} f = |\nabla f| = \sqrt 2$ and decreases in the dir. of $-\vec u = \langle \frac{1}{\sqrt 2}, -\frac{1}{\sqrt 2} \rangle$, $D_{\vec u} f = -\sqrt 2$

(No change in the dir. $\perp \vec u$)

- Gradients & Tangents to Level Curves

Surface $z = f(x,y)$ $\rightsquigarrow$ $f(x,y) = c$ is a level curve (in xy-plane)

If $f(x,y) = c$ along a smooth curve $\vec r(t)$:

$x = g(t)$, $y = h(t)$, then $f(g(t), h(t)) = c$

Differentiate w.r.t. t:

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\vec i + \frac{\partial f}{\partial y}\vec j\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dx}{dt}\vec i + \frac{dy}{dt}\vec j\right)}_{\frac{d\vec r}{dt} = \vec v \text{ tangent vector}} = 0 \quad \Rightarrow \quad \frac{d\vec r}{dt} \perp \nabla f$$

Thus, at every pt. (x_0, y_0) in the domain of a differentiable func. $f(x,y)$, the gradient ∇f is normal to the level curve going through (x_0, y_0)

Topographic maps:

Streams flow $\perp$ contours

↓

shows elevations

A downflowing stream reaches its destination in the fastest way, it must flow in the direction of $-\nabla f$, and

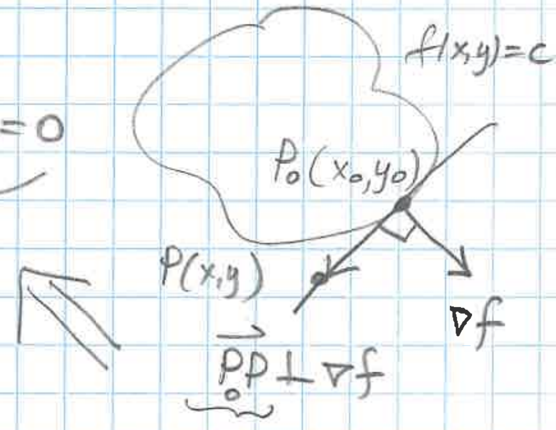
and these directions are $\perp$ the contours on the map (i.e., level curves).

(see Fig. 13.26, p. 713)

Also:

We can find equations for tangent lines from $\nabla f = \langle f_x, f_y \rangle$:

$$\underbrace{f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = 0}_{\text{Tangent to a level curve at } (x_0,y_0)}$$



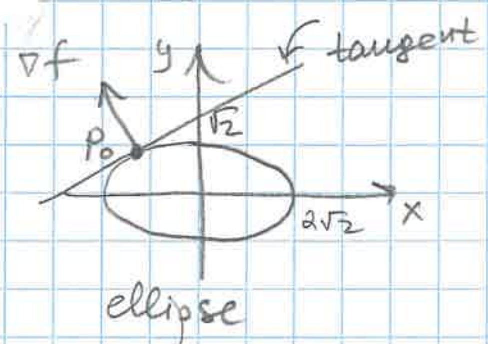
$\overrightarrow{P_0P} \perp \nabla f$

$\overrightarrow{P_0P} = \langle x-x_0, y-y_0 \rangle$

Example 3:

Let $z = \frac{x^2}{4} + y^2 = f(x,y)$

Consider the level curve $\frac{x^2}{4} + y^2 = 2$ going through $P_0(-2,1)$



$\nabla f = \langle \frac{x}{2}, 2y \rangle$

$\nabla f(-2,1) = \langle -1, 2 \rangle$ ← gradient at $(-2,1)$

Tangent line at $(-2,1)$:

$$(-1)(x+2) + (2)(y-1) = 0$$
$$-x + 2y - 4 = 0$$
$$x - 2y = -4.$$

More:

Algebra Rules for ∇f:

$$\nabla(f \pm g) = \nabla f \pm \nabla g$$
$$\nabla(kf) = k\nabla f \quad (k \in \mathbb{R})$$
$$\nabla(fg) = f\nabla g + g\nabla f$$
$$\nabla(f/g) = \frac{g\nabla f - f\nabla g}{g^2}$$

Note:

$D_{\vec{u}} f$ in $\mathbb{R}^3$: same, i.e.

for $f(x,y,z)$,

$$D_{\vec{u}} f = \nabla f \cdot \underbrace{\vec{u}}_{\substack{\text{unit vector} \\ \langle u_1, u_2, u_3 \rangle}}$$

$\nabla f = \langle f_x, f_y, f_z \rangle$