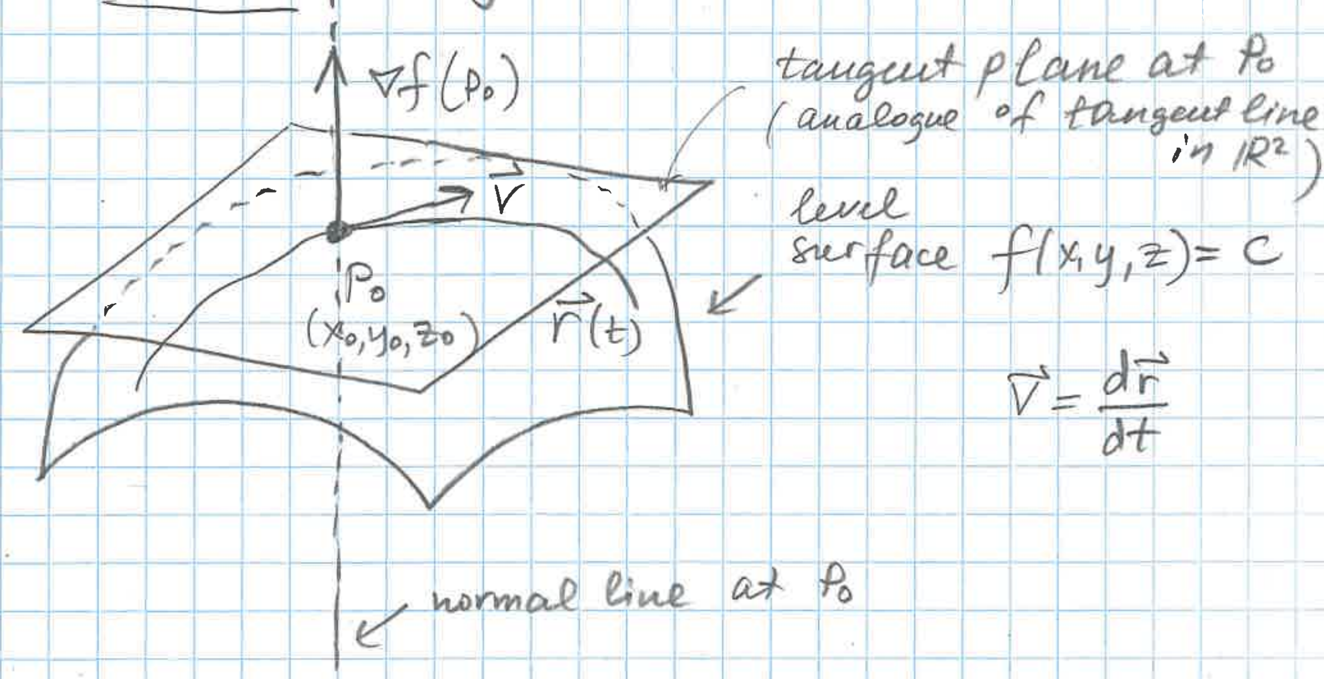
§ 13.6 Tangent Planes.



Let $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ be a smooth curve on the level surface $f(x,y,z)=C$

Then $\frac{d}{dt} f(\vec{r}(t)) = \frac{d}{dt} f(x(t), y(t), z(t))$

$$\overset{\text{Chain rule}}{=} \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \nabla f \cdot \underbrace{\frac{d\vec{r}}{dt}}_{=\vec{V}}$$

Since $f(x,y,z) = C$ then $\nabla f \cdot \frac{d\vec{r}}{dt} = 0 \Rightarrow$

$\nabla f \perp \frac{d\vec{r}}{dt} = \vec{v}$ for any $\vec{r}$ on the surface.

Let us define a plane through a pt. P_0 formed by all the tangent lines at P_0 normal to ∇f_{P_0}.

<u>Def</u>: The tangent plane at $P_0(x_0, y_0, z_0)$ on the level surface $f(x,y,z) = C$ of a differentiable func. f is the plane through P_0, normal to ∇f_{P_0} with the equation

(1) $\quad f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$

Def: The normal line of the surface $f(x,y,z)=C$ at P_0 is the line through P_0, parallel to ∇f_{P_0}.

The equations of the normal line are:

(2) $x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t$

Example 1: Find the tangent plane and normal line of the level surface $\underbrace{\frac{x^2}{4} + y^2 + \frac{z^2}{9}}_{f(x,y,z)} = 3$ (ellipsoid) at $P_0(-2,1,-3)$.

Solution:

$$\nabla f = \frac{x}{2}\vec{i} + 2y\vec{j} + \frac{2z}{9}\vec{k}$$

$$\nabla f(P_0) = -\vec{i} + 2\vec{j} - \frac{2}{3}\vec{k}$$

Tangent plane: $(-1)(x-(-2)) + 2(y-1) + (-\frac{2}{3})(z-(-3)) = 0$

$$\boxed{-x + 2y - \frac{2}{3}z = 6}$$

Normal line: $\boxed{\begin{aligned} x &= -2 - t \\ y &= 1 + 2t \\ z &= -3 - \tfrac{2}{3}t \end{aligned}}$

- Now suppose a surface is given by equ. $z = f(x,y)$. This is equivalent to $f(x,y) - z = 0$ which can be considered as the zero-level surface of func. $F(x,y,z) = f(x,y) - z$. Then

$$F_x = \frac{\partial}{\partial x}(f(x,y) - z) = f_x - 0 = f_x$$
$$F_y = \frac{\partial}{\partial y}(f(x,y) - z) = f_y - 0 = f_y$$
$$F_z = \frac{\partial}{\partial z}(f(x,y) - z) = 0 - 1 = -1$$

Then the tangent plane to $z=f(x,y)$ at $P_0(x_0,y_0)$ has the eqn:

(3) $f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)-(z-z_0)=0$

(compare to eqn. (1))

<u>Example 2</u>: Find the plane tangent to $z=f(x,y)=x^3y^4$ at $P_0(1,1,1)$:

$f_x = 3x^2y^4 \Rightarrow f_x(1,1) = 3$
$f_y = 4x^3y^3 \Rightarrow f_y(1,1) = 4$

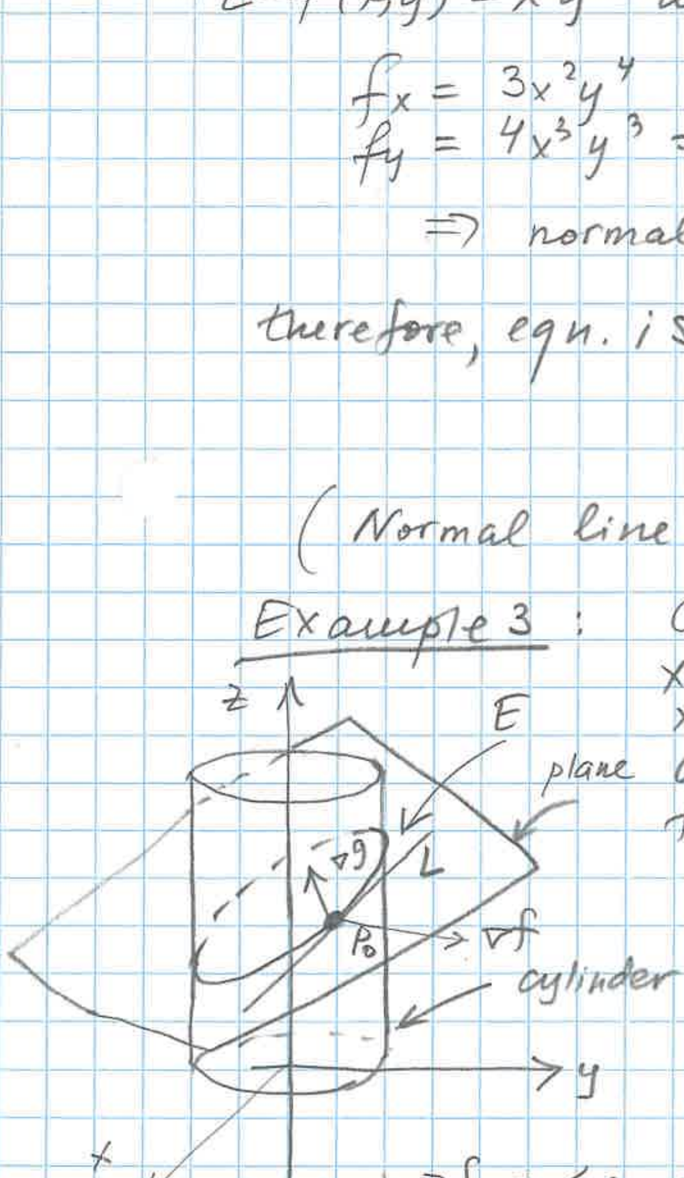
$\Rightarrow$ normal to plane is $\langle 3,4,-1\rangle$ and,

therefore, eqn. is $3(x-1)+4(y-1)-(z-1)=0$

$$\boxed{3x+4y-z=6}$$

(Normal line: $x=1+3t,\ y=1+4t,\ z=1-t$)

<u>Example 3</u>: Consider a cylinder $f(x,y,z)= x^2+y^2-2=0$ & a plane $g(x,y,z)= x+z-4=0$ that meet in an ellipse E. Find param. eqns for the line tangent to E at $P_0(1,1,3)$



<u>Solution</u>: (E: $x=\sqrt{2}\cos t,\ y=\sqrt{2}\sin t,\ z=4-\sqrt{2}\cos t$)

Tang. line to E, $L \perp \nabla f$ & ∇g

$\Rightarrow L \parallel \nabla f \times \nabla g$ at $P_0(1,1,3)$

$\nabla f = \langle 2x, 2y, 0\rangle,\ \nabla f(1,1,3) = \langle 2,2,0\rangle \Rightarrow$

$(\vec{n}=)\nabla g = \langle 1,0,1\rangle,\ \nabla g(1,1,3) = \langle 1,0,1\rangle$

$\nabla f \times \nabla g = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \langle 2,-2,-2\rangle \Rightarrow$

L has eqn's $x=1+2t,\ y=1-2t,\ z=3-2t$

• Linearization (briefly):

$z = f(x,y)$ can be complicated, and one may want to approximate them with simpler ones.

Recall: Given a pt. (x_0, y_0), if $f(x,y)$ is differentiable,

$$\underset{(\text{or } \Delta z)}{\Delta f} = f(x,y) - f(x_0,y_0) = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \underbrace{\text{error}}_{\varepsilon_1 \Delta x + \varepsilon_2 \Delta y}$$

$\Delta x = x - x_0$
$\Delta y = y - y_0$

$\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$

$$\Downarrow$$

$$\underbrace{f(x,y)}_{z} \approx \underbrace{\overbrace{f(x_0,y_0)}^{z_0} + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

$f(x,y) \approx L(x,y)$ is the standard linear approximation of f at (x_0, y_0)

(Recall: for $y = f(x)$, $f(x) \approx \underbrace{f(x_0) + f'(x_0)(x-x_0)}_{L(x)}$)

Geometrically: $z = L(x,y)$ is the plane tangent to $z = f(x,y)$ at (x_0, y_0).

So, $f(x,y) \approx L(x,y)$ is a tangent-plane approx.

(for $y = f(x)$, $f(x) \approx L(x)$ is a tangent-line approx.)

If $w = f(x,y,z)$ is a differentiable func. of x, y, z, then $f(x,y,z) \approx L(x,y,z)$ at a pt. $P_0(x_0, y_0, z_0)$, where

$$L(x,y,z) = f(P_0) + f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0)$$