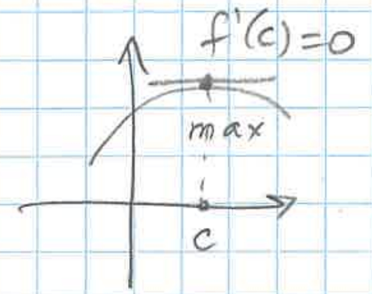
§13.7 Extreme Values & Saddle Points

Calc. 1:

$y = f(x)$



c - critical pt.
slope at c is 0.

$f(c) \geq f(x)$ for any x near c

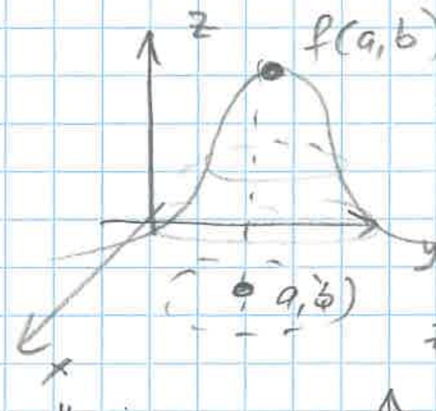
Similarly, for $z = f(x,y)$ we'll be looking for pts where the surface has a horizontal tangent plane $\longrightarrow$ $f_x = f_y = 0$ in equation $z = z_0 + f_x(x-x_0) + f_y(y-y_0)$

Def: (Local max/min)

Let $f(x,y)$ be defined on a region R containing the pt. (a,b)

1) $f(a,b)$ is a <u>local maximum</u> value of f if $f(a,b) \geq f(x,y)$ for all pts (x,y) in an open disk centered at (a,b)

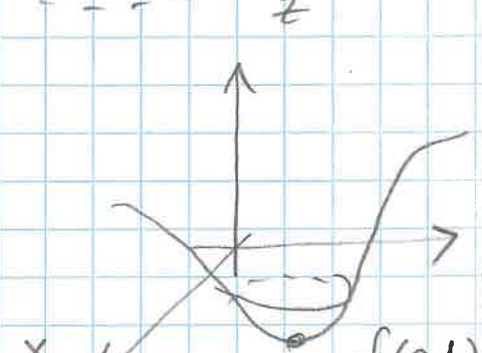
2) $f(a,b)$ is a <u>local minimum</u> value of f if $f(a,b) \leq f(x,y)$ for all pts (x,y) in an open disk centered at (a,b)



loc. max (no greater value nearby)

Loc. max: "mountain peaks"

Loc. min: "valley bottoms"



loc. min (no smaller value nearby)

Theorem (10): First Derivative Test for Local Extrema.

If $f(x,y)$ has a local min or max at (a,b) and if f_x & f_y exist there, then

$$f_x(a,b) = f_y(a,b) = 0.$$

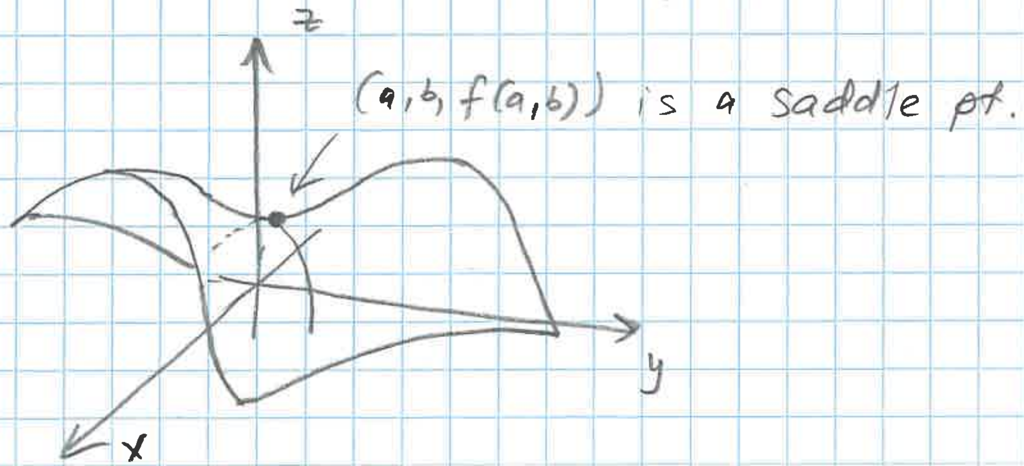
(proof in the text)

Def: An interior pt of the domain of f where both f_x, f_y are zero or do not exist is a critical pt. of f

- f can have extrema at critical pts and (or) boundary pts. (Not every critical pt. is a pt. of extremum though.)

Def: (Saddle pt)

A differentiable func. $f(x,y)$ has a saddle pt. at a critical pt. (a,b) if in every open disk centered at (a,b) there are pts where $f(x,y) > f(a,b)$ & pts where $f(x,y) < f(a,b)$. The pt. $(a,b,f(a,b))$ on the surface $z = f(x,y)$ is a saddle pt. of f.



Example 1: $f(x,y) = x^2 + y^2 - 2x - 6y + 14$
Find & identify local extrema

Domain of f: entire xy-plane (no boundary pts)
$f_x = 2x - 2$ & $f_y = 2y - 6$ exist everywhere.
Extrema can occur where

$$f_x = 2x - 2 = 0 \quad \& \quad f_y = 2y - 6 = 0$$
$$x = 1 \qquad\qquad y = 3$$

So, $(1,3)$ is a critical pt. Is $f(1,3) = 4$ max? min?

Check: $f(x,y) = x^2 - 2x + y^2 - 6y + 14$

$$= (x^2 - 2x + 1) - 1 + (y^2 - 6y + 9) - 9 + 14$$

$$= \underbrace{(x-1)^2}_{\geq 0} + \underbrace{(y-3)^2}_{\geq 0} + 4 \geq 4 \Rightarrow (1,3,4) \text{ is a loc. min}$$

Example 2: $f(x,y) = y^2 - x^2$
Find & identify local extrema

Domain of f: entire xy-plane (no boundary pts)

$f_x = -2x$, $f_y = 2y$ exist everywhere

Critical pt: $\left.\begin{array}{l} -2x = 0 \Rightarrow x = 0 \\ 2y = 0 \Rightarrow y = 0 \end{array}\right\} \Rightarrow (0,0)$,
Is $f(0,0) = 0$ max? min?

No, since:

along x-axis $(y=0)$ $f(x,y) = -x^2 < 0$, but
along y-axis $(x=0)$ $f(x,y) = y^2 > 0$

So, in an open disk centered at $(0,0)$ there pts where $f > 0$ & $f < 0 \Rightarrow$ no extremum at $(0,0)$, we have a saddle pt.

Again: having $f_x = f_y = 0$ at (a,b) does not guarantee having an extremum there.

In practice

Theorem (11) Second Derivative Test for Local Extrema

Suppose that $f(x,y)$, $f_x, f_y, f_{xx}, f_{yy}, f_{xy} = f_{yx}$ are continuous in a disk centered at (a,b). Then

1) f has a <u>loc. max</u> at (a,b) if $f_{xx} < 0$ &

$f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b)

2) f has a <u>loc. min</u> at (a,b) if $f_{xx} > 0$ &

$f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b)

3) f has a <u>saddle pt</u> at (a,b) if

$f_{xx} f_{yy} - f_{xy}^2 < 0$ at (a,b)

4) the test is inconclusive if $f_{xx} f_{yy} - f_{xy}^2 = 0$ at (a,b).

- Discriminant or Hessian of f:

$$H = f_{xx} f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

$H > 0$ or $H < 0$ or $H = 0$

$H > 0$: $f_{xx} < 0 \Downarrow$ loc. max; $f_{xx} > 0 \Downarrow$ loc. min

$H < 0 \Downarrow$ saddle pt

$H = 0 \Downarrow$ test fails.

Example 3 Find local extrema and saddle pts of $\underbrace{f(x,y) = x^4 + y^4 - 4xy + 1}$

defined in the entire plane, no boundary pts

$$f_x = 4x^3 - 4y$$
$$f_y = 4y^3 - 4x$$
$$f_{xx} = 12x^2$$
$$f_{yy} = 12y^2$$
$$f_{xy} \; (= f_{yx}) = -4$$

Critical pts: $f_x = 0$ & $f_y = 0$

$4x^3 - 4y = 0$ $\qquad$ $4y^3 - 4x = 0$

$\underbrace{x^3 = y}$ $\qquad$ $y^3 = x$

plug

$$(x^3)^3 = x$$
$$x^9 - x = 0$$
$$x(x^8 - 1) = 0$$
$$x(x^4-1)(x^4+1) = 0$$
$$x(x^2-1)(x^2+1)(x^4+1) = 0$$
$$x(x-1)(x+1)\underbrace{(x^2+1)}_{\neq 0}\underbrace{(x^4+1)}_{\neq 0} = 0$$

$\boxed{x = 0, \; x = \pm 1}$

3 pts to check: $(0,0)$, $(1,1)$, $(-1,-1)$

$f(0,0) = 1$

$f(1,1) = -1$

$f(-1,-1) = -1$

$$H = f_{xx} f_{yy} - f_{xy}^2 = 144x^2y^2 - 16$$

1) $H(0,0) = -16 < 0$ $\Leftarrow$

$\Rightarrow$ saddle pt at $(0,0)$

2) $H(1,1) = 144 - 16 > 0$ & $f_{xx}(1,1) = 12 > 0$

$\Rightarrow$ loc. min at $(1,1)$

3) $H(-1,-1) = 144 - 16 > 0$ & $f_{xx}(-1,-1) = 12 > 0$

$\Rightarrow$ loc. min at $(-1,-1)$

Read Ex. 3, 4, 5 pp. 733–734.

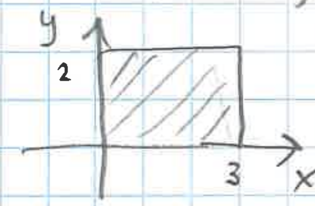
Absolute Maxima & Minima on Closed Bounded Regions.

① Find critical pts of f ($f_x = f_y = 0$) & evaluate f at them

② List boundary pts of f where f has local max/min

③ Look through the lists ① & ② for abs. min/max of f.

Example 4 Find absolute extrema of $f(x,y) = x^2 - 2xy + 2y$ on the rectangle
$D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$



Solution:

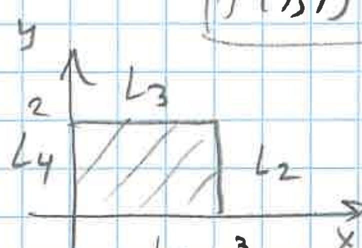
① Critical pts: $f_x = 2x - 2y$
$f_y = -2x + 2$

$f_x = 0$ | $f_y = 0$
$2x = 2y$ | $-2x + 2 = 0$
$x = y$ | $x = 1$

$\Rightarrow x = 1,\ y = x = 1 \Rightarrow$ $(1,1)$ is a crit. pt.

$\boxed{f(1,1) = 1}$

② Boundary pts:



Take one side at a time!

(i) L_1: $y = 0$
$f(x,0) = x^2$, $0 \le x \le 3$, $f(x,0)\uparrow$ on $[0,3] \Rightarrow$

min is $\boxed{f(0,0) = 0}$, max is $\boxed{f(3,0) = 9}$

(ii) L_2: $x = 3$
$f(3,y) = 9 - 6y + 2y = 9 - 4y$, $0 \le y \le 2$.

$f(3,y) \searrow$ on $[0,2] \Rightarrow$ min is $\boxed{f(3,2)=1}$
max is $\boxed{f(3,0)=9}$

(iii) L_3 : $y=2$

$f(x,2) = x^2 - 4x + 4$, $0 \le x \le 3$ min/max? not obvious

set $g(x) = f(x,2) = x^2-4x+4$

$\Rightarrow g'(x) = 2x-4 = 0 \Rightarrow x=2$ is a critical pt
$g''(x) = 2 > 0 \Rightarrow$ min at $x=2$

cr. pt: $g(2) = \boxed{f(2,2) = 0} \longrightarrow$ min

endpts $\begin{cases} g(0) = \boxed{f(0,2) = 4} \longrightarrow \text{max} \\ g(3) = f(3,2) = 1 \end{cases}$

(iv) L_4 : $x=0$

$f(0,y) = 2y$, $0 \le y \le 2$

$f(0,y) \nearrow$ on $[0,2] \Rightarrow$ min is $\boxed{f(0,0)=0}$
max is $\boxed{f(0,2)=4}$

3 List all values at cr. pts & bound. pts:

$f(1,1)=1$
$\boxed{f(0,0)=0} \longrightarrow$ abs. min
$\boxed{f(3,0)=9} \longrightarrow$ abs. max
$f(3,2)=1$
$\boxed{f(2,2)=0} \longrightarrow$ abs. min
$f(0,2)=4$