§ 13.8 Lagrange Multipliers.

Suppose $f(x,y,z)$ and $g(x,y,z)$ are differentiable functions and $\nabla g \neq 0$ when $g(x,y,z)=0$.

To find the local max & min values of f subject to the constraint $g(x,y,z)=0$ (if these exist), find the values of x, y, z, and λ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x,y,z) = 0$$

(For functions of two variables we have to find x, y, λ s.t. $\nabla f = \lambda \nabla g$ and $g(x,y)=0$.)

$\boxed{\lambda \text{ is a Lagrange multiplier}}$

(Read examples in the text & watch videos!)

Example: The temperature at a pt. (x,y) on a metal plate is

$$T(x,y) = 4x^2 - 4xy + y^2.$$

An ant on the plate walks around the circle of radius 5 centered at the origin. What are the highest & lowest temperatures encountered by the ant?

Solution: We want to find min/max of $T(x,y)$ subject to $x^2+y^2-25=0$ (equation of the circle).

$$\nabla T = \langle 8x-4y, -4x+2y \rangle, \quad \nabla g = \langle 2x, 2y \rangle$$

Using Lagrange multipliers:

$\nabla T = \lambda \nabla g \Rightarrow \langle 8x-4y, -4x+2y \rangle = \lambda \langle 2x, 2y \rangle$

and $x^2+y^2-25=0 \Rightarrow$ find x, y, λ satisfying

$$\left.\begin{array}{l} 8x-4y = 2\lambda x \\ -4x+2y = 2\lambda y \\ x^2+y^2-25=0 \end{array}\right\}$$

Express y from the second eqn: $y = -\frac{2x}{\lambda-1}$.

Then plug it into the first eqn:

$$8x - 4\left(-\frac{2x}{\lambda-1}\right) = 2\lambda x$$

$$8x + \frac{8x}{\lambda-1} = 2\lambda x$$

$$x\left(8 + \frac{8}{\lambda-1} - 2\lambda\right) = 0$$

$$x\left(8(\lambda-1) + 8 - 2\lambda(\lambda-1)\right) = 0$$

$$x\left(-2\lambda^2 + 10\lambda\right) = 0 \Rightarrow x=0 \text{ or } \lambda=0, \lambda=5$$

Case 1: $x=0 \Rightarrow y = -\frac{2\cdot 0}{\lambda-1} = 0$

Pt. $(0,0)$ is not on the circle $x^2+y^2-25=0$

Case 2: $\lambda=0 \Rightarrow y = -\frac{2x}{0-1} = 2x \Rightarrow$

$$x^2 + (2x)^2 = 25$$

$$5x^2 = 25$$

$$x^2 = 5 \Rightarrow x = \pm\sqrt{5} \Rightarrow y = \pm 2\sqrt{5}$$

Case 3: $\lambda = 5 \Rightarrow y = -\frac{2x}{5-1} = -\frac{x}{2} \Rightarrow$

$$x^2 + \left(-\frac{x}{2}\right)^2 = 25$$

$$x^2 + \frac{x^2}{4} = 25$$

$$5x^2 = 100$$

$$x^2 = 20 \Rightarrow x = \pm 2\sqrt{5} \Rightarrow y = \mp\sqrt{5}$$

Thus, we have pts from Cases 2 & 3:

$(2\sqrt{5}, -\sqrt{5})$, $(-2\sqrt{5}, \sqrt{5})$, $(\sqrt{5}, 2\sqrt{5})$, $(-\sqrt{5}, -2\sqrt{5})$.

By plugging these pts into $T(x,y)$, we find that

$$\left.\begin{aligned} T(2\sqrt{5}, -\sqrt{5}) &= 125^\circ \\ T(-2\sqrt{5}, \sqrt{5}) &= 125^\circ \end{aligned}\right\} \text{ max. value of } T(x,y) \text{ on circle } x^2+y^2=25$$

$$\left.\begin{aligned} T(\sqrt{5}, 2\sqrt{5}) &= 0^\circ \\ T(-\sqrt{5}, -2\sqrt{5}) &= 0^\circ \end{aligned}\right\} \text{ min. value of } T(x,y) \text{ on circle } x^2+y^2=25$$

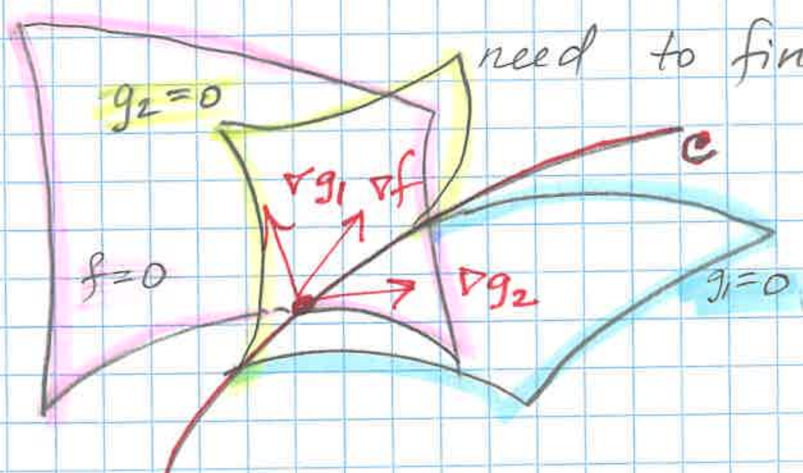
(See other examples in the text!)

- Lagrange multipliers with 2 constraints.

If we want to find extrema of $f(x,y,z)$ subject to constraints $g_1(x,y,z)=0$ & $g_2(x,y,z)=0$ then if ∇g_1 is not parallel to ∇g_2, we need 2 Lagrange multipliers, $\boxed{\lambda \text{ \& } \mu}$:

$$\underbrace{\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \quad g_1(x,y,z)=0, \quad g_2(x,y,z)=0}$$

need to find x, y, z, λ, μ satisfying these eqns



- $g_1=0$ & $g_2=0$ intersect in a curve C
- ∇f lies in the plane determined by ∇g_1 & ∇g_2 & $\nabla f \perp C$ (also, $\nabla g_1 \perp C$, $\nabla g_2 \perp C$)

(Read Example 5 p. 745)