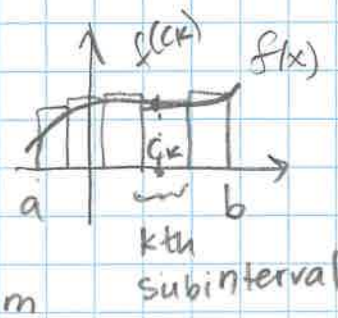
Chapter 14: Multiple Integrals

§ 14.1 Double & Iterated Integrals over Rectangles.

Calc. 2: $y = f(x)$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} f(c_k)\Delta x_k}_{\text{Riemann sum}}$$



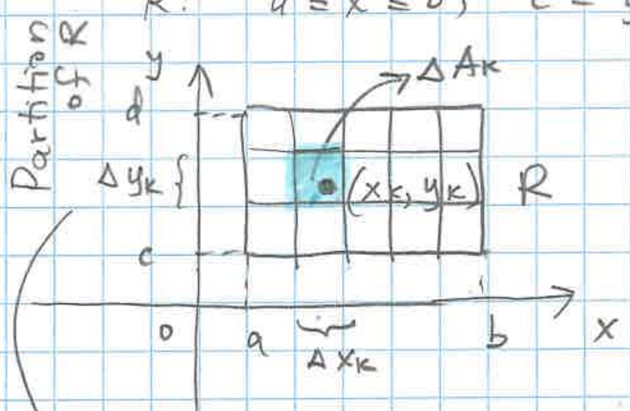
Recall:

$f(x) \geq 0 \Rightarrow$ area is $\int_a^b f(x)\,dx$

$f(x) \leq 0 \Rightarrow \int_a^b f(x)\,dx \leq 0$ is negative area

Double integrals:

$z = f(x,y)$ is defined on a rectangle R, i.e.
$R: \; a \leq x \leq b, \; c \leq y \leq d$



Subdivide R into n small rectangles with sides parallel to x- & y-axes (partition of R)

ΔA_k is the area of kth rectangle, $\Delta A_k = \Delta x_k \Delta y_k$

Form a Riemann sum over R:

$$S_n = \sum_{k=1}^{n} f(\underbrace{x_k, y_k}_{\text{pt in } k\text{th rectangle}})\,\Delta A_k$$

each rectangular box has height $f(x_k, y_k)$ (if $f \geq 0$) and the base area ΔA_k

$\Rightarrow S_n$ is the volume of the solid that consists of n boxes (if $f(x,y) \geq 0$)

If $\lim_{n\to\infty} S_n$ exists, then Volume $= \lim_{n\to\infty} S_n$

$(\Delta x_k, \Delta y_k \to 0)$ $\qquad$ $(f(x,y) \geq 0!)$

Def: $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$ is called the double integral of f over R, written as:

$$\iint_R f(x,y)\,dA \quad \text{or} \quad = \iint_R f(x,y)\underbrace{dx\,dy}_{dA}$$

(provided it exists) and f is called integrable.

Note: if f is continuous $\Rightarrow$ f is integrable.

Q: How do we calculate double integrals?

Fubini's Theorem (First Form)

If $f(x,y)$ is continuous throughout the rectangular region $R: a \le x \le b,\ c \le y \le d$, then

$$\iint_R f(x,y)\,dA = \underbrace{\int_c^d\int_a^b f(x,y)\,dx\,dy}_{} = \underbrace{\int_a^b\int_c^d f(x,y)\,dy\,dx.}_{}$$

says: can integrate in either order!

called iterated or repeated integrals

Example 1: Find volume under the plane $z = 4 - x - y$ over the region R: $0 \le x \le 2,\ 0 \le y \le 1$ in the xy-plane.

Solution: Volume is $\iint_R (4-x-y)\,dA$

slices of volume ⊥ to y-axis

$$= \int_0^1 \overbrace{\int_0^2 (4-x-y)\,dx}^{\text{fix } y}\,dy = \int_0^1 \left[4x - \frac{x^2}{2} - xy\right]_0^2 dy = \int_0^1 \left(8 - \frac{2^2}{2} - 2y\right) dy = \int_0^1 (6-2y)\,dy = \left[6y - y^2\right]_0^1 = \boxed{5}$$

or

slices of volume ⊥ to x-axis

$$= \int_0^2 \overbrace{\int_0^1 (4-x-y)\,dy}^{\text{fix } x}\,dx = \int_0^2 \left[4y - xy - \frac{y^2}{2}\right]_0^1 dx = \int_0^2 \left(4 - x - \frac{1}{2}\right) dx = \int_0^2 \left(\frac{7}{2} - x\right) dx = \left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^2 = \boxed{5}$$

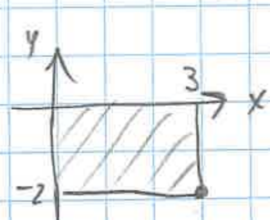
Example 2: Evaluate $\displaystyle\int_0^3\int_{-2}^0 (x^2y-2xy)\,dy\,dx$

$$\uparrow x \quad \uparrow y$$

$$= \int_0^3 \Big[\underbrace{\int_{-2}^0 (x^2y-2xy)\,dy}_{\text{fix } x} \Big]\,dx = \int_0^3 \Big[x^2\frac{y^2}{2} - 2x\frac{y^2}{2} \Big]_{-2}^{0} dx$$

$$= \int_0^3 \Big[0 - \Big(\frac{x^2(-2)^2}{2} - x(-2)^2 \Big) \Big]\,dx = \int_0^3 (-2x^2+4x)\,dx$$

$$= \Big[-2\frac{x^3}{3} + \frac{4x^2}{2} \Big]_0^3 = -\frac{2\cdot 27}{3} + \frac{4\cdot 9}{2} = -18+18 = 0$$

R: 

e.g., $f(3,-2)<0$

Note: x^2y-2xy is both negative & positive over R $\Rightarrow$ our zero integral is the sum of volume under the surface (where $z \geq 0$) and negative volume (where $z \leq 0$)