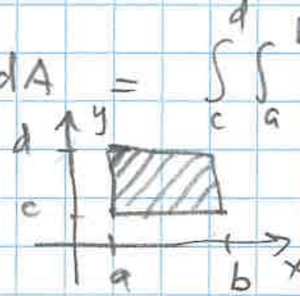
§14.2 Double Integrals over General Regions.

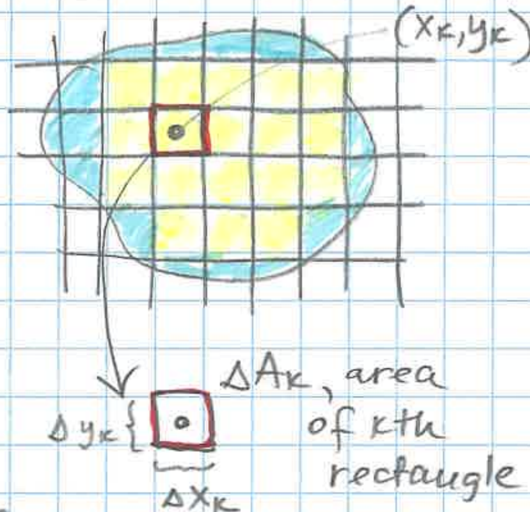
Recall: $\iint\limits_R f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy \underset{\text{Fubini's}}{=} \int_a^b\int_c^d f(x,y)\,dy\,dx$

(f - cont. on R)

R: 

rectangular region in xy-plane

Q: What if our region is not a rectangle?

R: 

- Create a rectangular grid
- Use only rectangles lying completely inside the boundary
- Let # rectangles $\to \infty$ (with $\Delta x_k, \Delta y_k \to 0$) $\Rightarrow$ the "good" rectangles will "fill" R.

(Partition: n rectangles)

- Pick a pt (x_k, y_k) in the k-th cell.
- Form a Riemann sum $S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$

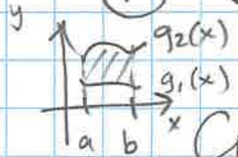
& take $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \sum_{k=1}^{n} f(x_k,y_k)\,\Delta A_k = \underbrace{\iint\limits_R f(x,y)\,dA}_{\text{double integral of } f \text{ over } R}$

($\|P\| \to 0$) norm of partition, largest Δy_k or Δx_k

Volumes: if $\boxed{f(x,y) \geq 0}$ & continuous over R, then the volume of the solid region between R & the surface $z = f(x,y)$ is

$$\boxed{V = \iint\limits_R f(x,y)\,dA}$$

(as before)

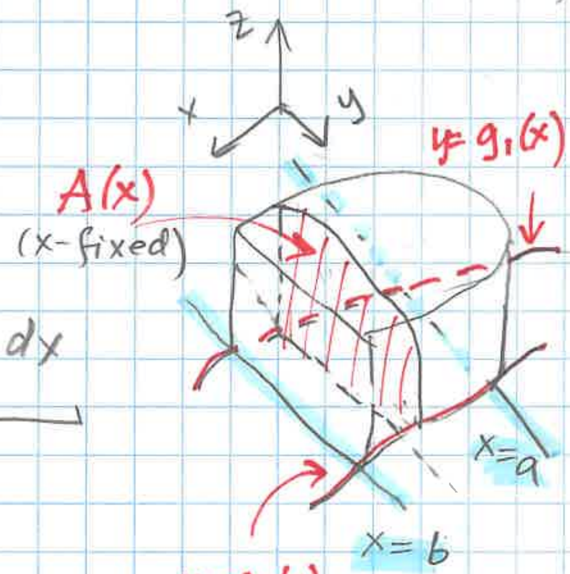


① Suppose R: $a \le x \le b$

$g_1(x) \le y \le g_2(x)$ (g_1, g_2 are cont. on $[a,b]$)

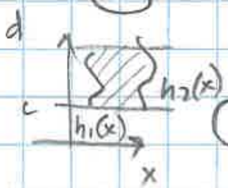
Cross-sectional area:

$$A(x) = \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy$$



$\Downarrow$

$$V = \int_a^b A(x)\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\underline{dy\,dx}$$

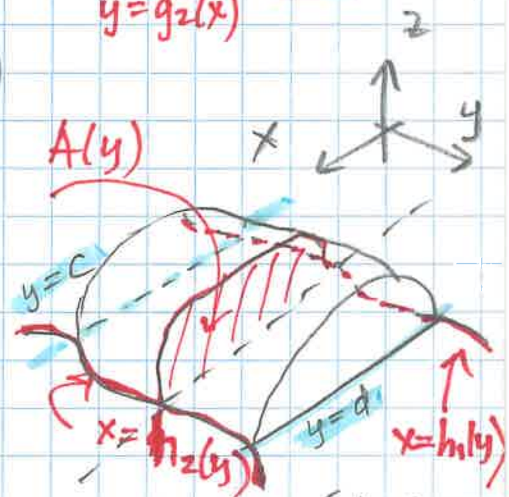


② Suppose R: $c \le y \le d$

$h_1(y) \le x \le h_2(y)$

Cross-sectional area:

$$A(y) = \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dx$$



$\Downarrow$

$$V = \int_c^d A(y)\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,\underline{dx\,dy}$$

(h_1, h_2 are cont on $[c,d]$)

$\Downarrow$

<u>Theorem 2</u> Fubini's Theorem (<u>Stronger Form</u>):

Let f be continuous over a region R

① If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, and g_1, g_2 are continuous on $[a,b]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

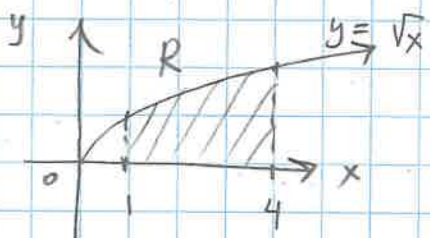
2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, and h_1, h_2 are continuous on $[c,d]$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

Examples:

1. Sketch the region R in xy-plane & compute the integral:

$$\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}}\,dy\,dx = \int_1^4 \left[\frac{3}{2} e^{y/\sqrt{x}} \cdot \sqrt{x}\right]_0^{\sqrt{x}} dx$$



$$= \int_1^4 \left(\frac{3}{2} e\sqrt{x} - \frac{3}{2}\sqrt{x}\right) dx$$

Note $\frac{3}{2} e^{y/\sqrt{x}} > 0$

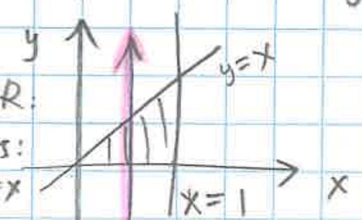
$$= \frac{3}{2}(e-1) \int_1^4 x^{1/2}\,dx$$

$\Rightarrow$ the integral gives the volume of the solid between R & the surface $z = \frac{3}{2} e^{y/\sqrt{x}}$

$$= \frac{3}{2}(e-1)\left[\frac{x^{3/2}}{3/2}\right]_1^4$$

$$= (e-1)(4^{3/2} - 1) = \boxed{7(e-1)}$$

2. Calculate $\iint_R \frac{\sin x}{x}\,dA$ where R is the triangle in the xy-plane bounded by the x-axis, lines $y = x$ and $x = 1$.



y limits: $y = 0$ to $y = x$

x limits: $x = 0$ to $x = 1$

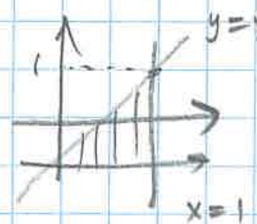
$$\iint_R \frac{\sin x}{x}\,dA = \int_0^1 \int_0^x \frac{\sin x}{x}\,dy\,dx =$$

$$= \int_0^1 \left[y \frac{\sin x}{x} \right]_0^x dx = \int_0^1 \sin x \, dx = -\cos 1 + 1 \approx 0.46$$

Let us reverse the order of integration:

$$\iint_R \frac{\sin x}{x} \, dA = \int_0^1 \underbrace{\int_y^1 \frac{\sin x}{x} \, dx}_{} \, dy \longrightarrow \text{problem.}$$

cannot be evaluated analytically (no simple antiderivative)



$\begin{cases} x \text{ limits: } x = y \text{ to } x = 1 \\ y \text{ limits: } y = 0 \text{ to } y = 1 \end{cases}$

→ So, if your first choice of order does not work, try the other one!

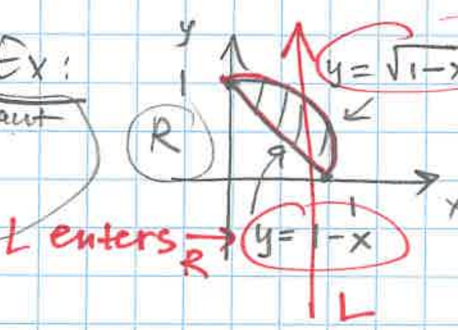
Limits of integration:

- "$dydx$" → using vertical cross-sections:

  1) sketch → 2) find y-limits: use vertical line L through R in the direction of increasing y, find y's where L enters & leaves → 3) find x-limits, s.t. they include all such lines L.

Ex:

1st quadrant
$x^2+y^2=1$
$x+y=1$



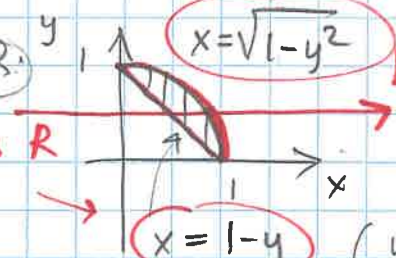
$$\iint_R f(x,y) \, dA = \int_0^1 \int_{1-x}^{\sqrt{1-x^2}} f(x,y) \, dy \, dx$$

- "$dxdy$" → using horizontal cross-sections:

  1) sketch → 2) find x-limits: use horizontal line L through R in the direction of increasing x, find x's where L enters and leaves → 3) find y-limits, s.t.

they include all such lines L.

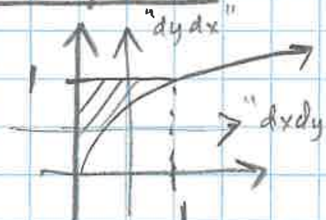
Ex. R:



$$\iint_R f(x,y)\,dA = \int_0^1\int_{1-y}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$$

More Examples:

3 R:



$y=\sqrt{x}$ (or $x=y^2$)

$$\iint_R \sqrt{3+2y^3}\,dA =$$

$$= \int_0^1\int_{\sqrt{x}}^1 \sqrt{3+2y^3}\,dy\,dx \overset{\text{or}}{=} \underbrace{\int_0^1\int_0^{y^2} \sqrt{3+2y^3}\,dx\,dy}_{\text{easier}}$$

$$= \int_0^1 \left[x\sqrt{3+2y^3}\right]_0^{y^2} dy = \int_0^1 y^2\sqrt{3+2y^3}\,dy$$

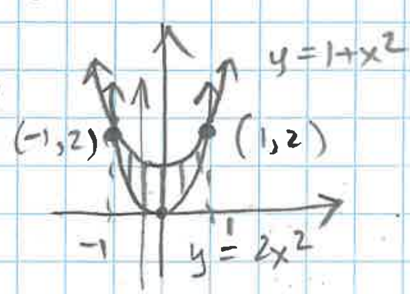
u-sub: $u = 3+2y^3$
$du = 6y^2 dx$
$u(0) = 3$
$u(1) = 5$

$$= \int_3^5 \frac{1}{6}\sqrt{u}\,du = \frac{1}{6}\,\frac{2u^{3/2}}{3/2}\Bigg]_3^5 =$$

$$= \frac{1}{9}\left(5\sqrt{5} - 3\sqrt{3}\right)$$

4 Evaluate $\iint_R (x+2y)\,dA$ where R is defined by $y = 2x^2$ & $y = 1+x^2$

R:



Pts of intersection:
$2x^2 = 1+x^2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$
$\Rightarrow (1,2)$ & $(-1,2)$

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Integral?

$$\iint_R (x+2y)\,dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx = \dots = \boxed{\frac{32}{15}}$$

using vertical cross-sections

Other way? No! (Need to split R)

- Properties of Double Integrals:

If $f(x,y)$ & $g(x,y)$ are continuous over R, then:

① $\iint_R c\,f(x,y)\,dA = c\iint_R f(x,y)\,dA$ (any number c)

→ Constant Multiple Rule

② $\iint_R (f(x,y) \pm g(x,y))\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$

→ Sum / Difference Rule

③ Domination:

(a) $\iint_R f(x,y)\,dA \geq 0$ if $f(x,y) \geq 0$ on R

(b) $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$ if $f(x,y) \geq g(x,y)$ on R

④ Additivity:

$$\iint f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

if $R = R_1 \cup R_2$

R: