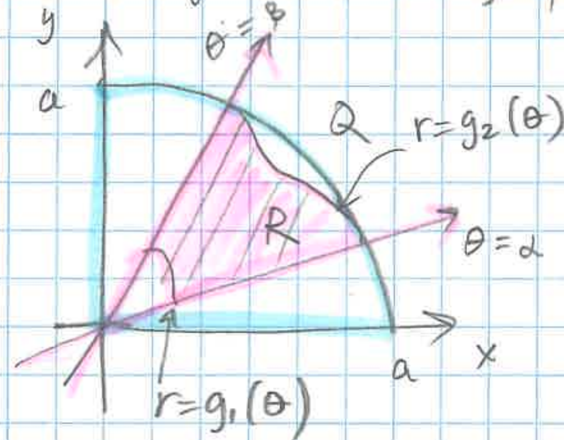
§14.4 Double Integrals in Polar Form.

Sometimes it is easier to evaluate a double integral using polar coordinates.

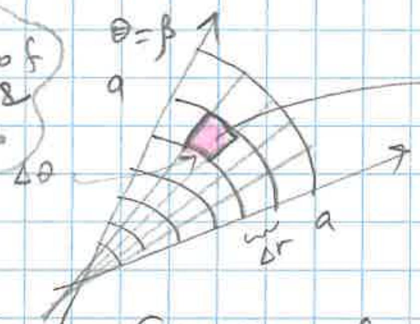


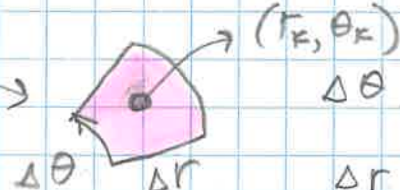
Region R (shaded) lies in region Q defined by:

$$\underbrace{\{(r,\theta) \mid 0 \le r \le a,\ \alpha \le \theta \le \beta\}}_{\text{fan-shaped region } Q,\ \text{so-called "polar rectangle"}}$$

Let us cover Q (and R inside it) with a grid:

Grid of arcs & rays





$$\Delta\theta = \frac{\beta-\alpha}{\ell}$$

$$\Delta r = \frac{a}{m}$$

Circles of radii $\Delta r, 2\Delta r, \ldots, m\Delta r$, and rays $\theta=\alpha, \alpha+\Delta\theta, \alpha+2\Delta\theta, \ldots, \theta = \alpha+\ell\Delta\theta=\beta$

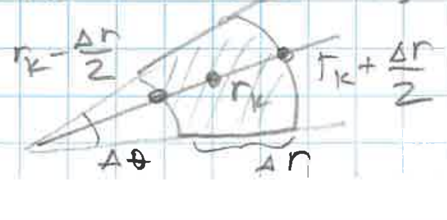
Thus, Q is partitioned into small polar rectangles of areas ΔA_k; choose those that lie inside R

Now form a Riemann sum

$$S_n = \sum_{k=1}^{n} f(\underbrace{r_k, \theta_k}_{\text{pt from a } k\text{th cell}}) \Delta A_k \xrightarrow[n\to\infty]{} \iint_R f(r,\theta) \underbrace{dA}_{?}$$

It can be shown that $\boxed{\Delta A_k = r_k (\Delta r)(\Delta\theta)}$ for r_k chosen to be an average of the radii of the inner and outer arcs bounding ΔA_k.

(see p. 774)



Thus, $\lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{n} f(r_k, \theta_k) \underbrace{r_k \Delta r_k \Delta \theta_k}_{\Delta A_k}$

$(dA = r\,dr\,d\theta)$

$$= \iint_R f(r,\theta)\, r \, dr\, d\theta$$

$\boxed{\text{do not forget this } r!}$

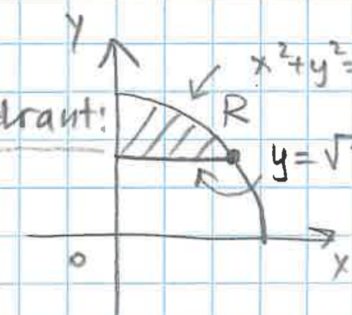
$$\underset{\substack{\text{by a version} \\ \text{of Fubini's} \\ \text{Theorem}}}{=} \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta) \underbrace{r\,dr\,d\theta}_{dA \text{ in polar coord's}}$$

Limits of Integration:

- Usually integrate w.r.t. r first, then w.r.t. θ.

(1) Sketch and label:

1st quadrant:

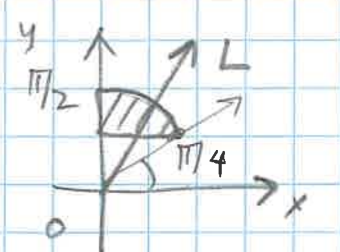


$x^2+y^2=4 \iff r=2$ (PC)

$y=\sqrt{2} \iff r\sin\theta = \sqrt{2}$ (PC)

(2) Find r-limits: draw a ray L from O to the direction of increasing r through R.

L enters R at $r = \frac{\sqrt{2}}{\sin\theta} = \sqrt{2}\csc\theta$ and leaves R at $r=2$

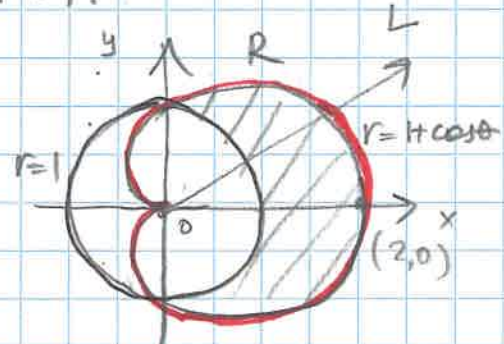


(3) Find θ-limits: min & max values for θ that bound R.

Thus, $\iint_R f(r,\theta)\,dA = \int_{\pi/4}^{\pi/2} \int_{\sqrt{2}\csc\theta}^{2} f(r,\theta)\,r\,dr\,d\theta$

$\left(\frac{1}{\sin\theta} = \csc\theta\right)$

Example 1: Find limits of integration for $\iint_R f(r,\theta)\,dA$ if R is the area inside the cardioid $r = 1+\cos\theta$ & outside the circle $r=1$.



θ	$r = 1+\cos\theta$
0	2
$\pi/3$	$3/2$
$\pi/2$	1
$2\pi/3$	$1/2$
π	0

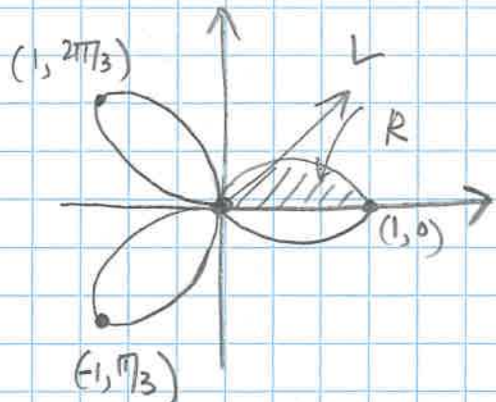
symm. about x-axis

$$\iint_R f(r,\theta)\,dA = \int_{-\pi/2}^{\pi/2} \int_{1}^{1+\cos 2\theta} f(r,\theta)\,r\,dr\,d\theta$$

- Area in Polar Coordinates:

$$A = \iint_R \underbrace{r\,dr\,d\theta}_{dA} \qquad (f(r,\theta) = 1 \text{ again!})$$

Example 2: Find the area enclosed by one petal of the rose $r = \cos 3\theta$ using double integration. (Recall: Ch. 10 uses single integral in polar coord's) $A = \int_\alpha^\beta \frac{1}{2} r^2\,d\theta$



θ	$r = \cos 3\theta$
0	1
$\pi/6$	0
$\pi/3$	-1
$\pi/2$	0
$2\pi/3$	1
$\frac{5\pi}{6}$	0
π	-1

(Using symmetry)

One petal area is $2 \cdot \iint_R dA$

$$= 2 \cdot \int_0^{\pi/6} \int_0^{\cos 3\theta} r\,dr\,d\theta$$

$$= 2 \int_0^{\pi/6} \left[\frac{r^2}{2}\right]_0^{\cos 3\theta} d\theta = \int_0^{\pi/6} \cos^2 3\theta\,d\theta = \int_0^{\pi/6} \frac{1+\cos 6\theta}{2}\,d\theta = \boxed{\pi/12}$$

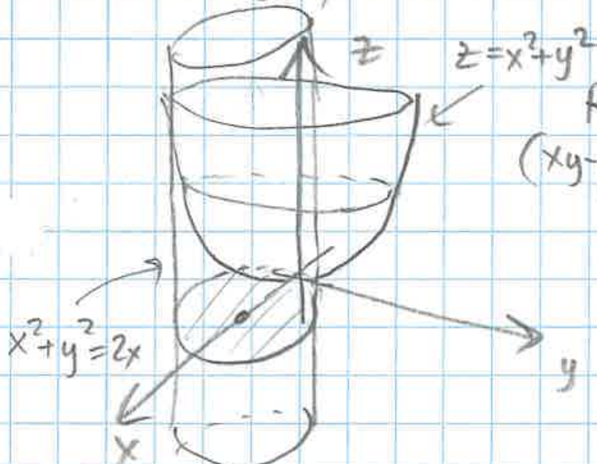
- Changing Cartesian Integrals to Polar Integrals.

$$\iint\limits_R f(x,y)\,\underbrace{dA}_{\text{in cart. coord's}} = \iint\limits_{R'} f(\underbrace{r\cos\theta}_{x}, \underbrace{y\sin\theta}_{y})\,\underbrace{r\,dr\,d\theta}_{\substack{dA \\ \text{in polar}}}$$

G is R described in polar coord's.

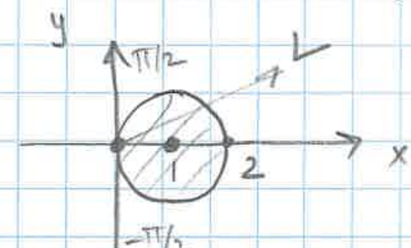
(See Examples 3-5 pp. 776-777)

Example: Find the volume of the solid that lies under paraboloid $z = x^2+y^2$, above the xy-plane and inside the cylinder $x^2+y^2=2x$



R: $x^2+y^2 = 2x \Rightarrow r^2 = 2r\cos\theta$
(xy-plane) $\Rightarrow r = 2\cos\theta$

$(x-1)^2+y^2 = 1$ circle of radius 1 centered at (1,0)



Region in xy-plane in polar coordinates:

$$R' = \left\{ (r,\theta) \mid 0 \le r \le 2\cos\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \right\}$$

$$V = \iint\limits_R (x^2+y^2)\,dA = \iint\limits_{R'} r^2\,\underbrace{r\,dr\,d\theta}_{dA}$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta =$$

$$= 4\int_{-\pi/2}^{\pi/2} \cos^4\theta\,d\theta = 4\int_{-\pi/2}^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta =$$

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$$= \int_{-\pi/2}^{\pi/2} \left(1 + 2\cos 2\theta + \underbrace{\frac{1}{2}\left(1+\cos 4\theta\right)}_{\cos^2 2\theta} \right) d\theta$$

$$= \left[\theta + \sin 2\theta + \frac{\theta}{2} + \frac{\sin 4\theta}{8} \right]_{-\pi/2}^{\pi/2} = \boxed{\frac{3\pi}{2}}$$