§ 14.5 Triple Integrals in Rectangular Coordinates.

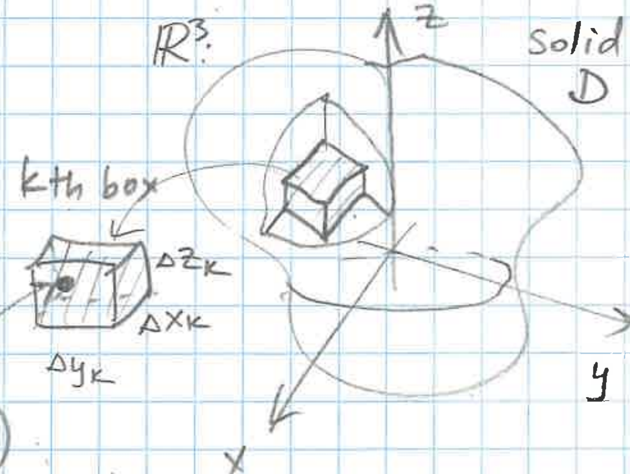
Used to calculate volumes of 3D shapes, average value of a function over a 3D region, in study of vector fields and fluid flows.

- Let $F(x,y,z)$ be defined on a closed bounded region D in space.

1) Break D into rectangular boxes (cells) by using planes parallel to the coordinate planes $x=0$, $y=0$, $z=0$

2) Suppose n such boxes are inside D with kth box with dimensions $\Delta x_k, \Delta y_k, \Delta z_k$,

$$\Rightarrow \Delta V_k = \Delta x_k \Delta y_k \Delta z_k$$



3) Choose a pt. (x_k, y_k, z_k) in the kth box and form

$$S_n = \sum_{k=1}^{n} F(x_k,y_k,z_k)\Delta V_k \longrightarrow \lim_{n\to\infty} S_n =$$

$(\|P\| \to 0)$ ↳ max of $\Delta x_k, \Delta y_k, \Delta z_k$ (over all k)

$$\underbrace{= \iiint_D F(x,y,z)\, dV}$$

If $\lim_{n\to\infty} S_n$ exists, then this is the triple integral of F over D; F is integrable over D.

- Volume of a region in space is given by

$$V = \underbrace{\iiint_D dV}_{\approx \sum_k \text{ of all } \Delta V_k} \qquad (F(x,y,z) = 1)$$

<u>Q</u>: How do we evaluate triple integrals?

we apply a 3D-version of Fubini's Theorem, i.e. use <u>3 repeated single integrations</u>.

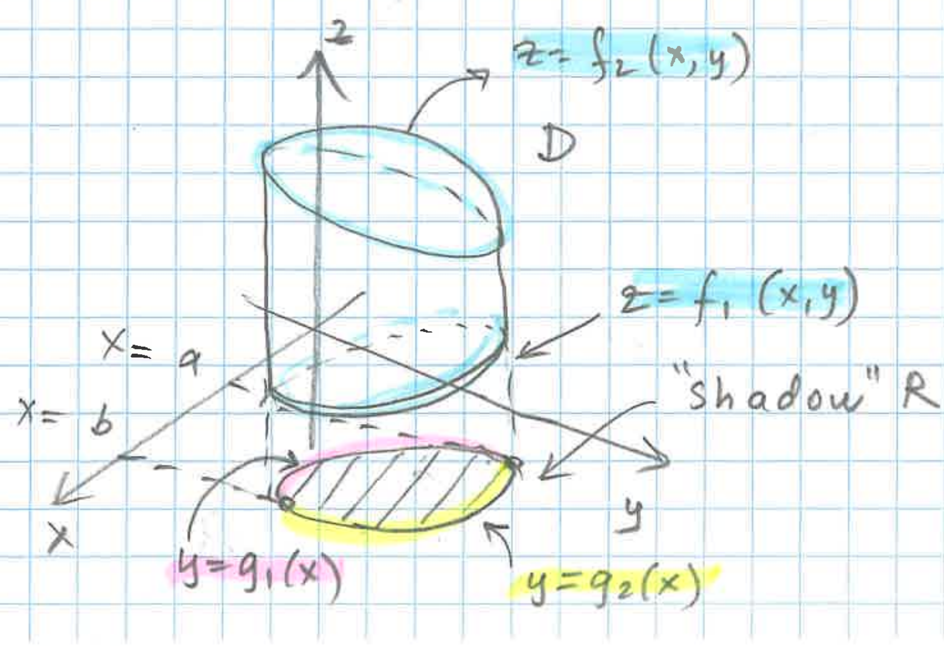
$$\iiint_D F(x,y,z)\, dV \longrightarrow$$ (almost always) integrate

(1) w.r.t. z
(2) w.r.t. y
(3) w.r.t. x

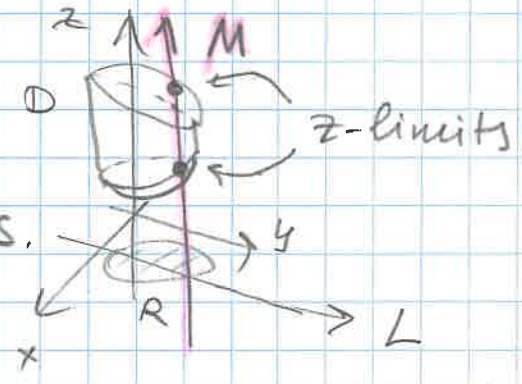
$\Big\}\ dz\,dy\,dx$

① Sketch region D (in 3D) w/ its "shadow" R in 2D (projection on the xy-plane)

Label the upper & lower bounding surfaces of D & upper & lower bounding curves of R



(2) <u>z-limits</u>: Draw a line M through D, parallel to z-axis. As $z \nearrow$, M enters D at $z = f_1(x,y)$ & leaves at $z = f_2(x,y)$.



(3) <u>y-limits</u>: Draw a line L through R parallel to y-axis. L enters R at $y = g_1(x)$ & leaves R at $y = g_2(x)$

(4) <u>x-limits</u> = limits that include all lines L through R // y-axis, $x=a$ to $x=b$.

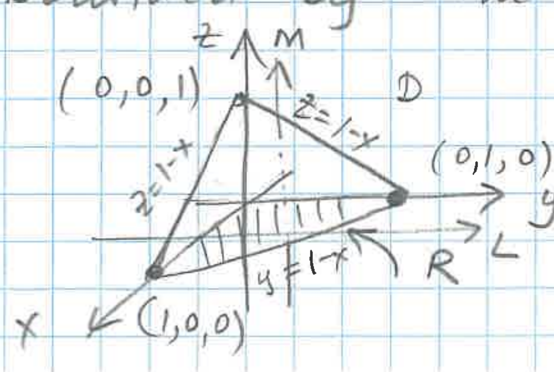
Thus, $$\iiint_D F(x,y,z)\,dV = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x,y,z)\,dz\,dy\,dx$$

• Average value of a function in 3D over a region D:

$$\boxed{\text{Aver. value of } F \text{ over } D = \frac{1}{\text{Volume of } D} \iiint_D F\,dV}$$

• <u>Read</u> examples 1-4 pp. 782-785!

• <u>Example 1</u>: Evaluate $\iiint_D z\,dV$, where D is the solid tetrahedron bounded by the 4 planes $x=0$, $y=0$, $z=0$ and $\underline{x+y+z=1}$.



D: $x=0 \Rightarrow y+z=1 \Rightarrow z=1-y$
$y=0 \Rightarrow x+z=1 \Rightarrow z=1-x$
$z=0 \Rightarrow x+y=1 \Rightarrow y=1-x$

$R: \{x+y=1,\ x=0,\ y=0\}$

$$\iiint_D z\,dV = \int_{x=0}^{x=1}\int_{y=0}^{1-x}\int_{z=0}^{z=1-x-y} z\,dz\,dy\,dx$$

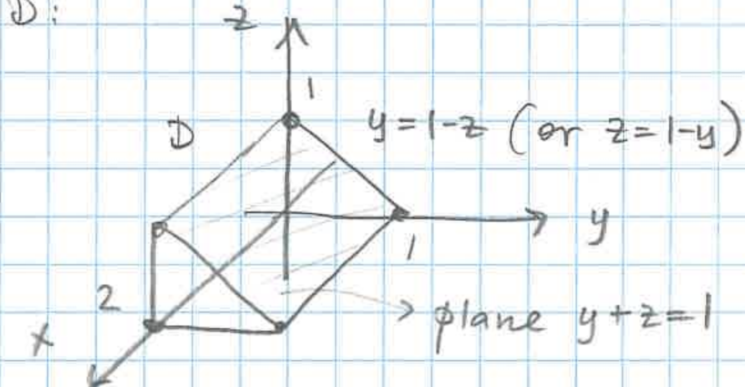
$$= \int_0^1\int_0^{1-x}\left[\frac{z^2}{2}\right]_0^{1-x-y} dy\,dx = \int_0^1\int_0^{1-x} \frac{(1-x-y)^2}{2}\,dy\,dx$$

$\left(\begin{array}{c}u\text{-sub:}\\ u = 1-x-y \\ du = -dy\end{array}\right)$

$$= \int_0^1 \left[-\frac{(1-x-y)^3}{2\cdot 3}\right]_0^{1-x} dx$$

$$= \frac{1}{6}\int_0^1 (1-x)^3\,dx = \frac{1}{6}\left[-\frac{(1-x)^4}{4}\right]_0^1 = \boxed{\frac{1}{24}}$$

Example 2: Each of the following integrals gives the volume of the solid D:



a) $\int_0^1\int_0^{1-z}\int_0^2 dx\,dy\,dz$

b) $\int_0^1\int_0^2\int_0^{1-z} dy\,dx\,dz$

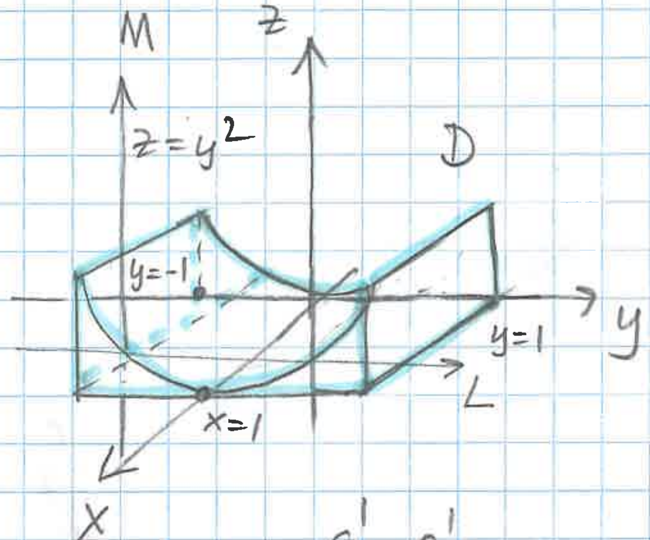
c) $\int_0^1\int_0^2\int_0^{1-y} dz\,dx\,dy$

d) $\int_0^1\int_0^{1-y}\int_0^2 dx\,dz\,dy$

e) $\int_0^2\int_0^1\int_0^{1-z} dy\,dz\,dx$

f) $\int_0^2\int_0^1\int_0^{1-y} dz\,dy\,dx$

Example 3: Find the volume of the solid between the cylinder $z = y^2$ & the xy-plane that is bounded by $\underbrace{x=0,\ x=1,\ y=-1,\ y=1}_{\text{planes}}$.



$$V = \iiint_D dV = \int_0^1 \int_{-1}^1 \int_0^{y^2} dz\,dy\,dx$$

$$= \int_0^1 \int_{-1}^1 \left[z\right]_0^{y^2} dy\,dx$$

$$= \int_0^1 \int_{-1}^1 y^2\,dy\,dx = \int_0^1 \left[\frac{y^3}{3}\right]_{-1}^1 dx$$

$$= \int_0^1 \left[\frac{1}{3} - \left(-\frac{1}{3}\right)\right] dx = \left[\frac{2}{3}x\right]_0^1 = \boxed{2/3}$$