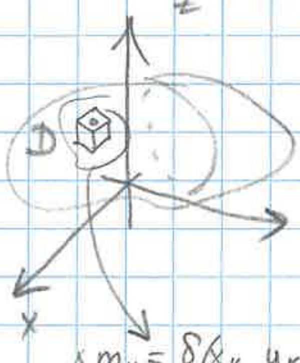
§ 14.6 Moments & Centers of Mass. ①

(no moments of inertia → optional reading.)



← Object occupying a region D in space

If $\delta(x,y,z)$ is the density (mass per unit volume) of the object, then

$\Delta m_k = \delta(x_k, y_k, z_k)\Delta V_k$ $\quad \iiint_D \delta(x,y,z)\,dv$ = mass of the object.

Why? Partition the object into n mass elements Δm_k, then

$$\underline{\text{mass}} \quad M = \lim_{n \to \infty} \sum_{k=1}^{n} \underbrace{\delta(x_k, y_k, z_k)\Delta V_k}_{\Delta m_k} = \iiint_D \delta\,dV$$

- The first moments of the mass D about the coordinate planes are:

$$M_{yz} = \iiint_D x\,\delta\,dV, \quad M_{xz} = \iiint_D y\,\delta\,dv, \quad M_{xy} = \iiint_D z\,\delta\,dV$$

→ Physics: moments measure the tendency of an object to twist/rotate about an axis. Also: shows balance of the object about different axes.

<u>Center of mass:</u> (or center of gravity)

- Unique pt. in an object /system which can be used to describe the object's responce to forces.

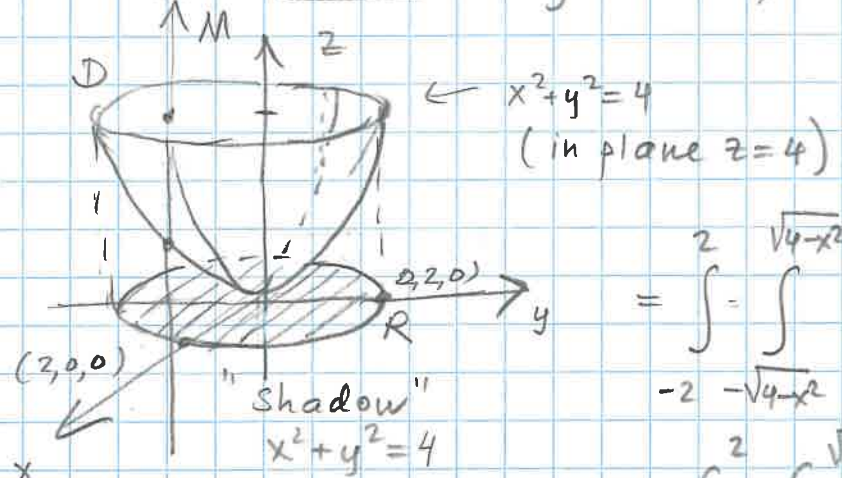
$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}$$

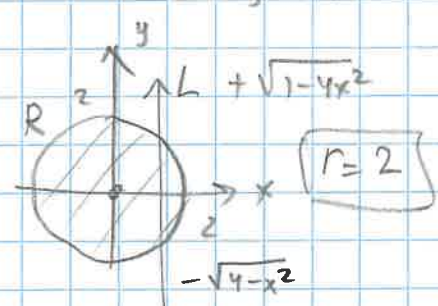
- If $\delta = 1$ then the center of mass $(\bar{x}, \bar{y}, \bar{z})$ is called the <u>centroid</u> of the object.

<u>In 2D</u>:

$$\begin{cases} M = \iint_R \delta \, dA \quad w/ \ \delta = \delta(x,y) \\ M_y = \iint_R x \delta \, dA, \quad M_x = \iint_R y \delta \, dA \\ \bar{x} = \dfrac{M_y}{M}, \quad \bar{y} = \dfrac{M_x}{M} \end{cases}$$

<u>Example</u>: Find the center of mass of a solid of constant density $\delta = 1$ bounded <u>below</u> by the paraboloid $z = x^2 + y^2$ and <u>above</u> by the plane $z = 4$.





$$M = \iiint_D \overset{\shortparallel}{\delta} \, dV$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{4} dz\,dy\,dx$$

$$= \underbrace{\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - \underbrace{(x^2+y^2)}_{r}) \overbrace{dy\,dx}^{r\,dr\,d\theta}}_{\text{easier to do in polar form!}}$$

$$= \int_0^{2\pi} \int_{r=0}^{r=2} (4 - r^2)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \underbrace{\left[\frac{4r^2}{2} - \frac{r^4}{4} \right]_0^2}_{4} d\theta = \int_0^{2\pi} 4\,d\theta = \left[4\theta\right]_0^{2\pi} = \underbrace{\boxed{8\pi}}_{\text{mass}}$$

Centroid : note that we have symmetry about z-axis $\Rightarrow$ in symmetrical systems, the center of mass lies on the symmetry axis!

So, $\bar{x} = \bar{y} = 0 \Rightarrow$ find only $\bar{z} = \dfrac{M_{xy}}{M}$

$$M_{xy} = \underbrace{\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}}_{R \,\to\, \text{turn to polar form } R'} \int_{x^2+y^2}^{4} z \, dz\, dy\, dx \quad (\delta = 1) = \iint_{R'} \left[\frac{z^2}{2}\right]_{x^2+y^2}^{4} dA$$

$$= \frac{1}{2} \iint_{R'} \left[16 - (x^2+y^2)^2\right] dA = \frac{1}{2} \int_0^{2\pi} \int_0^{2} \left[16 - r^4\right] r\, dr\, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^{2} (16r - r^5)\, dr\, d\theta = \int_0^{2\pi} \frac{32}{3}\, d\theta = \boxed{\frac{64\pi}{3}} = M_{xy}$$

So, $\bar{z} = \dfrac{\frac{64\pi}{3}}{8\pi} = \boxed{\dfrac{8}{3}}$ and

the centroid is the pt. $(0, 0, 8/3)$.