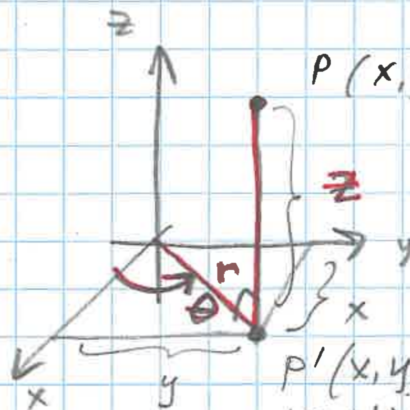
§14.7 Triple Integrals in Cylindrical & Spherical Coordinates.

(Easier to calculate for shapes like cones, cylinders, spheres, etc.!)

• Cylindrical Coordinates:



$P(x,y,z) = P(\underbrace{r,\theta,z})$ cylindrical coord's (ordered triple)

{ r & θ are polar coord's for P'
z is the rectangular vertical coordinate

$P'(x,y,0) = P'(r,\theta,0)$
vertical projection of P onto xy-plane

• Equations relating (x,y,z) & (r,θ,z) are:

$$\boxed{\begin{array}{ll} x = r\cos\theta & x^2+y^2=r^2 \\ y = r\sin\theta & \tan\theta = y/x \\ z = z & \end{array}}$$

Examples: in cylindrical coord's

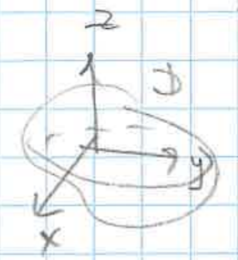
1) $r=5$ is a cylinder of radius 5 (along z-axis)

2) $\theta = \dfrac{\pi}{4}$ is a plane containing line $\theta = \pi/4$ and z-axis

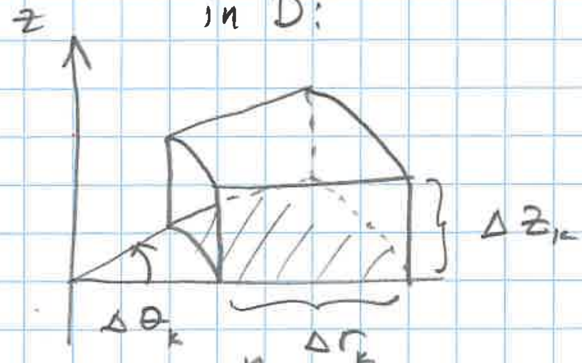
3) $z=1$ is a plane $\perp$ z-axis through $(0,0,1)$

• Triple Integrals in Cylindrical Coord's:

Partition a 3D region D into n small cylindrical wedges (analog of rectangular boxes):



in D:



$$\Delta V_k = \underbrace{\Delta z_k}_{\text{height}} \underbrace{r_k \Delta r_k \Delta \theta_k}_{\Delta A_k}$$

(ΔA_k seen in polar double integrals)

$$\Rightarrow S_n = \sum_{k=1}^{n} f(r_k, \theta_k, z_k) \Delta V_k$$

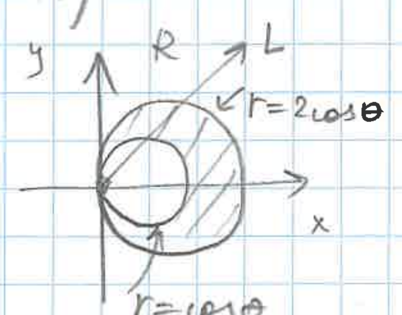
$$= \sum_{k=1}^{n} f(r_k, \theta_k, z_k) \Delta z_k \, r_k \Delta r_k \Delta \theta_k \xrightarrow[n \to \infty]{}$$

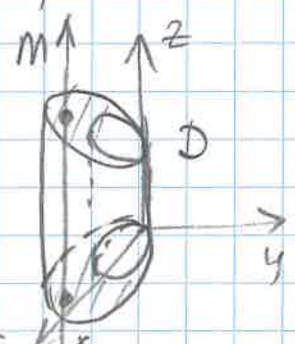
$$\iiint_D f(r, \theta, z) \, dV = \iiint_D f(r, \theta, z) \underbrace{dz \, r \, dr \, d\theta}_{dV}$$

(If $f = 1 \Rightarrow V = \iiint_D dV = \iiint_D dz \, r \, dr \, d\theta$)

Example 1: Set up the integral of $f(r, \theta, z)$ over D, where D is the solid right cylinder whose base is the region between the curves $r = \cos\theta$ & $r = 2\cos\theta$ ($z = 0$) & whose top lies in the plane $z = 3 - y$.

Base:



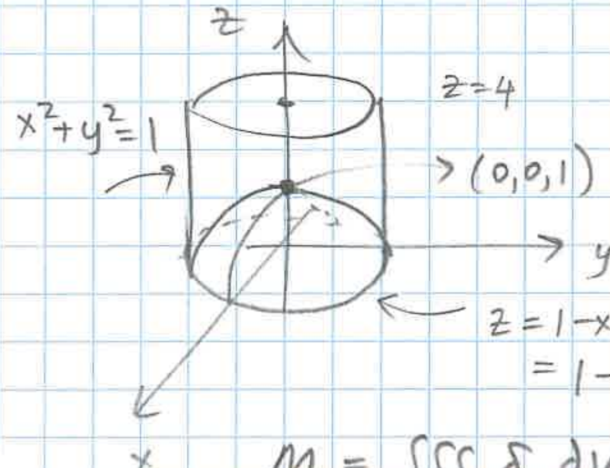


$z = 3 - y$
$= 3 - r\sin\theta$

$$\iiint_D f \, dV = \int_{\theta=-\pi/2}^{\theta=\pi/2} \int_{r=\cos\theta}^{r=2\cos\theta} \int_{z=0}^{3 - r\sin\theta} f(r, \theta, z) \, r \, dz \, dr \, d\theta$$

Example 2: Solid E lies within the cylinder $x^2+y^2=1$, below the plane $z=4$ & above the paraboloid $z=1-x^2-y^2$. The density at any point (in E) is proportional to the distance from the point to the z-axis (axis of the cylinder). Find the mass of E.

Solution:



$z=1-x^2-y^2$
$=1-r^2$

$$\delta(x,y,z) = \underset{\text{constant}}{k}\,\underbrace{\sqrt{(x-0)^2+(y-0)^2+(z-z)^2}}_{\substack{\text{distance}\\ \text{from } (x,y,z)\\ \text{to z-axis}}}$$

$$\Rightarrow \delta = k\underbrace{\sqrt{x^2+y^2}}_{r} = kr$$

$$M = \iiint_E \delta\, dV = \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 \underbrace{kr}_{\delta}\,\underbrace{dz\,r\,dr\,d\theta}_{dV}$$

$$= \int_0^{2\pi}\int_0^1 \Big[kr^2 z\Big]_{1-r^2}^{4}\, dr\,d\theta = \int_0^{2\pi}\int_0^1 kr^2\left(4-(1-r^2)\right) dr\,d\theta$$

$$= k\int_0^{2\pi}\int_0^1 (3r^2+r^4)\,dr\,d\theta = k\int_0^{2\pi}\left[r^3+\frac{r^5}{5}\right]_0^1 d\theta$$

$$= k\int_0^{2\pi}\left[1+\frac{1}{5}\right]d\theta = k\left[\frac{6}{5}\theta\right]_0^{2\pi} = \boxed{k\,\frac{12\pi}{5}}$$

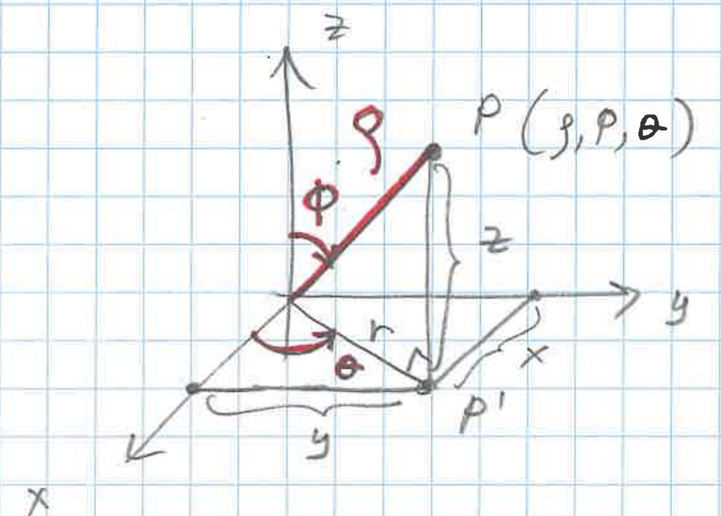
• Spherical Coordinates:

represent a pt. P in space by ordered triples (ρ, φ, θ), where:

1) ρ is the distance from P to the origin, $\boxed{\rho \geq 0}$

2) φ is the angle that $\overrightarrow{OP}$ makes with the positive z-axis, $\boxed{0 \leq \varphi \leq \pi}$

3) θ is the angle from cylindrical (polar) coord's.

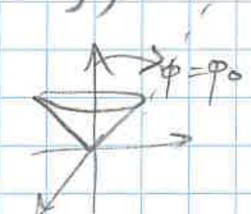


On Earth maps:

θ - meridian of a pt.
φ - latitude
ρ - elevation above the surface

Examples: 1) $\rho = a$ (constant) is sphere of radius a (φ, θ vary)

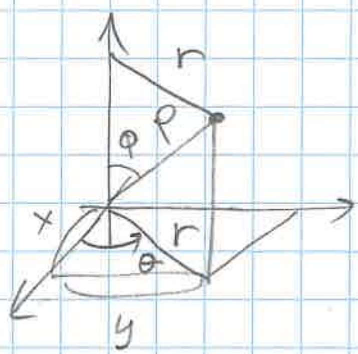
2) $\varphi = \varphi_0$ (fixed) $\rightarrow$ single cone
($\varphi = \frac{\pi}{2}$ is the xy-plane)

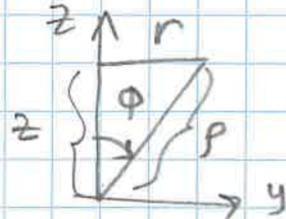


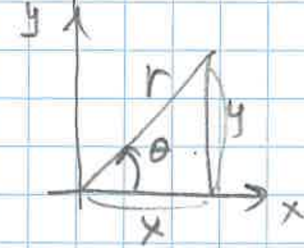
3) $\theta = \theta_0$ (fixed) $\longrightarrow$ half plane
(φ, $\rho \geq 0$ vary)

- Equations relating spherical, Cartesian, and cylindrical coord's:

$$\boxed{\begin{array}{ll} r = \rho\sin\varphi & x = r\cos\theta = \rho\sin\varphi\cos\theta \\ z = \rho\cos\varphi & y = r\sin\theta = \rho\sin\varphi\sin\theta \\ \rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2} & \end{array}}$$



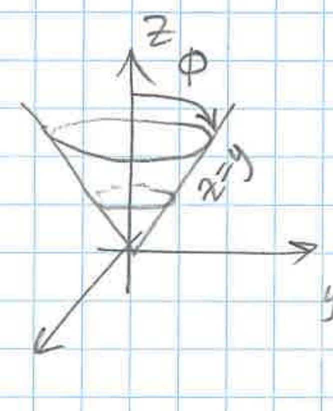




Example 1: Find a spherical coordinates equation for the cone $z = \sqrt{x^2+y^2}$ $(z \geq 0)$

- Geometrically:

Cone cuts yz-plane $(x=0)$ along
$z = \sqrt{0+y^2} = y \geq 0$
→ angle with z-axis is
$\varphi = \pi/4$



So, $\boxed{\varphi = \frac{\pi}{4}}$

$\rho \geq 0$ and θ varies unrestricted.

- Algebraically: $z = \underbrace{\sqrt{x^2+y^2}}_{\parallel} \overset{\text{and}}{=} \rho\cos\varphi$

$$\sqrt{(\rho\sin\varphi\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2}$$

So, $z = \sqrt{\rho^2 \sin^2\varphi(\cos^2\theta + \sin^2\theta)} = \rho \underbrace{\sin\varphi}_{\geq 0}$ (6)

$0 \leq \varphi \leq \pi$

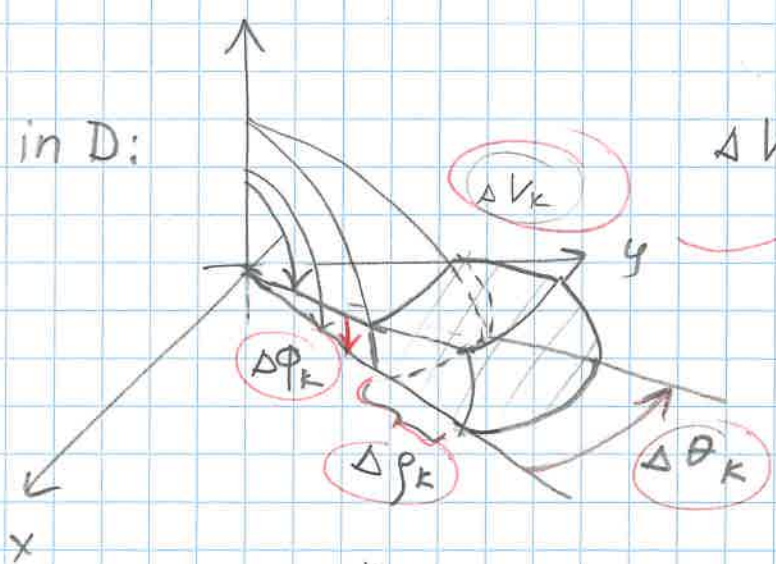
& $z = \rho\cos\varphi \Rightarrow$

$\sin\varphi = \cos\varphi \Rightarrow \varphi = \frac{\pi}{4} \quad (0 \leq \varphi \leq \pi)$

Q: How do we integrate in spherical coord's?

Partition a 3D shape D into n spherical wedges: $\iiint_D f(\rho, \varphi, \theta)\, \underbrace{dV}_{?}$

in D:



$\Delta V_k = \rho_k^2 \sin\varphi_k \Delta\rho_k \Delta\varphi_k \Delta\theta_k$

So, $S_n = \sum_{k=1}^{n} \underbrace{f(\rho_k, \varphi_k, \theta_k)}_{\text{pt chosen inside the kth wedge}} \underbrace{\rho_k^2 \sin\varphi_k \Delta\rho_k \Delta\varphi_k \Delta\theta_k}_{\Delta V_k}$

$\Rightarrow S_n \xrightarrow[n\to\infty]{} \iiint_D f(\rho, \varphi, \theta) \underbrace{\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta}_{dV}$

(Volume is $V = \iiint_D dV = \iiint_D \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$)

<u>To integrate:</u> (1) sketch D $\rightarrow$ (2) find ρ-limits (ray from O through D): ρ from $\rho = g_1(\varphi, \theta)$ to $\rho = g_2(\varphi, \theta)$. $\rightarrow$

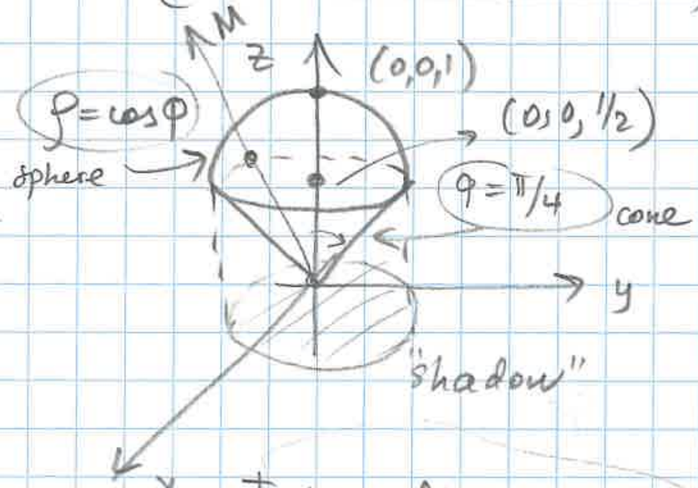
(3) φ-limits: given by φ_{min} & φ_{max} that rays in (2) make with positive z-axis.

$\rightarrow$ (4) θ limits: ray in xy-plane that sweeps over the shadow gives $\theta = \alpha$ to β.

$$\iiint_D f(\rho, \varphi, \theta)\, dV = \int_\alpha^\beta \int_{\varphi_{min}}^{\varphi_{max}} \int_{g_1(\varphi,\theta)}^{g_2(\varphi,\theta)} f(\rho, \varphi, \theta)\, \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

<u>Example 2:</u> Use spherical coord's to find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$.

("Ice cream cone")



$z = \sqrt{x^2+y^2} \Leftrightarrow \varphi = \pi/4$

$x^2+y^2+z^2 = z$ is
$x^2+y^2+\left(z-\frac{1}{2}\right)^2 = 1/4$ —
sphere of radius $1/2$ centered at $(0,0,1/2)$

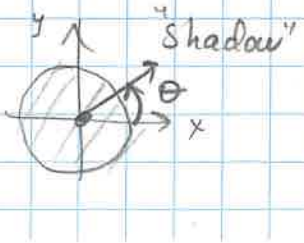
Also: $\underbrace{\rho^2}_{x^2+y^2+z^2} = \underbrace{\rho\cos\varphi}_{z}$

$\Rightarrow \rho = \cos\varphi$

Intersection:
if $x = 0 \Rightarrow$
$z = y$ & $y^2+z^2 = z$

$\Rightarrow z = 0$ or $z = 1/2$

$\Rightarrow$ in yz-plane: $(0,0,0)$ & $(0, 1/2, 1/2)$



So, $V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\varphi} (1)\, \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

$(\theta) \quad (\varphi)$

$$= \int_0^{2\pi} \int_0^{\pi/4} \left[\frac{\rho^3}{3} \sin\varphi \right]_0^{\cos\varphi} d\varphi \, d\theta$$

$$= \int_0^{2\pi} \underbrace{\int_0^{\pi/4} \frac{\cos^3\varphi}{3} \sin\varphi \, d\varphi}_{} \, d\theta$$

u-sub: $u = \cos\varphi$

$du = -\sin\varphi \, d\varphi$

$u(0) = 1$

$u(\pi/4) = \sqrt{2}/2$

$$= \int_0^{2\pi} \underbrace{\int_1^{\sqrt{2}/2} \left(-\frac{1}{3}\right) u^3 \, du}_{} \, d\theta = -\frac{1}{3} \int_0^{2\pi} \left[\frac{u^4}{4} \right]_1^{\frac{\sqrt{2}}{2}} d\theta$$

$$= -\frac{1}{3} \int_0^{2\pi} \left(\frac{1}{16} - \frac{1}{4} \right) d\theta = \frac{1}{16} \left[\theta \right]_0^{2\pi} = \boxed{\frac{\pi}{8}}$$