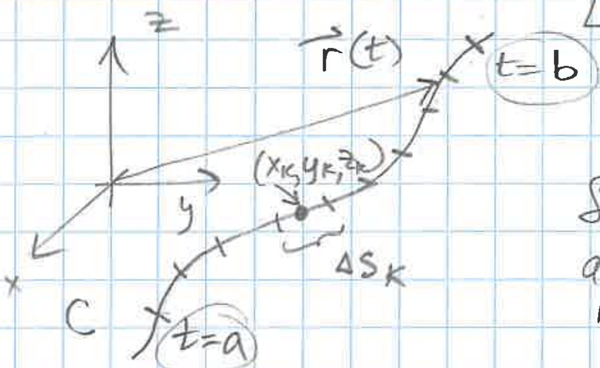
Chapter 15. Integrals & Vector Fields.

§15.1. Line Integrals.

A new kind of integral. (Need to remember how to parameterize curves!) Line integrals are used for finding the masses of thin rods (wires) lying along a curve in the plane or space or the work done by a force acting along a curve.



Let $\vec{r}(t) = \underbrace{g(t)}_{x}\,\vec{i} + \underbrace{h(t)}_{y}\,\vec{j} + \underbrace{k(t)}_{z}\,\vec{k}$

$a \le t \le b$ (curve C)

Suppose $f(x,y,z)$ is a function we want to integrate over $\vec{r}(t)$ (f - composite func.)

(1) Partition C into n sub-arcs with lengths ΔS_k

(2) Choose a pt. (x_k, y_k, z_k) in the kth sub-arc.

(3) Form $S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta S_k \xrightarrow[(\Delta S_k \to 0)]{n \to \infty}$

$$\underbrace{\int_C f(x,y,z)\,ds}$$

the line integral of f over C (provided the limit exists).

Q: How do we evaluate $\int_C f\,\overset{?}{\widetilde{ds}}$?

If C is smooth (i.e., $\vec{v} = \frac{d\vec{r}}{dt} \neq 0$ and $\frac{d\vec{r}}{dt}$ is continuous) and if f is continuous on C then $\underbrace{s(t)}_{\text{arc length}} = \int_a^t \underbrace{|\vec{v}(\tau)|}_{\text{speed (length of } \vec{v})} d\tau$

$\Rightarrow \frac{ds}{dt} \underset{(FTC)}{=} |\vec{v}(t)| \Rightarrow \underbrace{ds}_{\text{differential}} = |\vec{v}(t)|\,dt$

$\left(= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \right)$

Thus, $\boxed{\int_C f(x,y,z)\underbrace{ds}_{|\vec{v}(t)|dt} = \int_a^b f(\underbrace{g(t)}_{x}, \underbrace{h(t)}_{y}, \underbrace{k(t)}_{z})\,|\vec{v}(t)|\,dt}$

Note: 1) This integral is independent of choice of parameterization

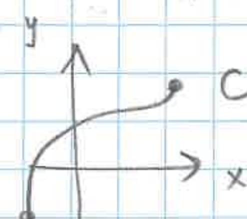
2) If $f = 1 \Rightarrow \int_C ds = \underline{\text{length of } C}$

3) Need to find parameterization of C!

In the Plane: (similar)

$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$

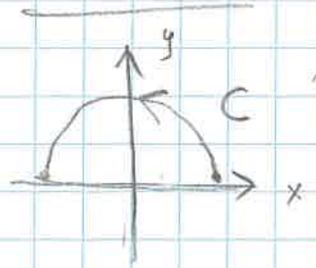
$a \le t \le b$



$\Rightarrow \int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\,|\vec{v}(t)|\,dt$

Example 1: Evaluate $\int_C (2 + x^2 y)\,ds$ where C is the upper half of the unit circle.

Solution:



$x^2 + y^2 = 1 \Rightarrow x = \cos t,\ y = \sin t,\ 0 \le t \le \pi$

$\Rightarrow |\vec{v}(t)| = \sqrt{\sin^2 t + \cos^2 t} = 1$

So, $$\int_C (2+x^2y)\,ds = \int_0^{\pi} (2+\cos^2 t\cdot \sin t)(1)\,dt$$

$$= \underbrace{\int_0^{\pi} 2\,dt}_{2\pi} + \underbrace{\int_0^{\pi} \cos^2 t \sin t\,dt}_{\substack{\text{let } u=\cos t \\ du = -\sin t\,dt \\ u(0)=1,\ u(\pi)=-1}}$$

$$= 2\pi + \int_{-1}^{1} u^2\,du = 2\pi + \left[\frac{u^3}{3}\right]_{-1}^{1} = \boxed{2\pi + 2/3}$$

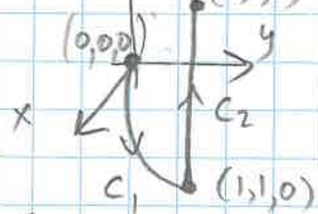
(See Example 1, p. 822)

- <u>Additivity</u>: If C is made by joining n smooth curves $C_1, \dots, C_n$, end to end, then

$$\int_C f\,ds = \int_{C_1} f\,ds + \dots + \int_{C_n} f\,ds$$

<u>Example 2</u>: Integrate $f(x,y,z) = x+\sqrt{y}-z^2$ over the path from $(0,0,0)$ to $(1,1,1)$ given by
C_1 : $\vec{r}_1(t) = t\vec{i} + t^2\vec{j}$, $0 \le t \le 1$, and then by
C_2 : $\vec{r}_2(t) = \vec{i} + \vec{j} + t\vec{k}$, $0 \le t \le 1$.

<u>Solution</u>: $\vec{V}_1(t) = \vec{i} + (2t)\vec{j} + 0\vec{k} \Rightarrow |\vec{V}_1| = \sqrt{1+4t^2}$
$\vec{V}_2(t) = 0\vec{i} + 0\vec{j} + 1\vec{k} \Rightarrow |\vec{V}_2| = 1$



(Path of integration: $C_1 \cup C_2$)

$$\int_{C=C_1\cup C_2} (x+\sqrt{y}-z^2)\,ds = \underbrace{\int_0^1 (t+\sqrt{t^2}-0^2)\sqrt{1+4t^2}\,dt}_{=\int_{C_1} f\,ds} + \underbrace{\int_0^1 (1+\sqrt{1}-t^2)(1)\,dt}_{=\int_{C_2} f\,ds}$$

$$\int_{C_1} f\,ds = \int_0^1 2t\sqrt{1+4t^2}\,dt \qquad \left(\text{use } u\text{-sub } u = 1+4t^2\right)$$

$$= \frac{1}{6}\left(5\sqrt{5}-1\right)$$

$$\int_{C_2} f\,ds = \int_0^1 (2-t^2)\,dt = 5/3$$

Therefore, $\int_C f\,ds = \left(\frac{5}{6}\sqrt{5} - \frac{1}{6}\right) + \frac{5}{3} = \boxed{\frac{5}{6}\sqrt{5} + \frac{3}{2}}$

- Mass & Center of Mass Formulas

(for coil springs, wires, thin rodes lying along a smooth curve C in space)

- Mass: $M = \int_C \delta\,ds$, $\delta = \delta(x,y,z)$ is the density func.

- Coordinates of the center of mass $(\bar{x}, \bar{y}, \bar{z})$:

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}, \quad \text{where}$$

$$\underbrace{M_{yz} = \int_C x\,\delta\,ds, \quad M_{xz} = \int_C y\,\delta\,ds, \quad M_{xy} = \int_C z\,\delta\,ds}_{\text{First moments.}}$$

Example 3: Find the mass of a wire lying along the curve $\vec{r}(t) = (t^2-1)\vec{j} + 2t\vec{k}$, $0 \le t \le 1$, with density $\delta = 3/2\,t$.

Solution: $M = \int_C \delta\,ds = \int_0^1 \frac{3}{2}t\left(2\sqrt{t^2+1}\right)dt$

$\vec{v}(t) = 0\vec{i} + (2t)\vec{j} + 2\vec{k}$ (use $u = t^2+1$)

$|\vec{v}| = \sqrt{4t^2+4} = 2\sqrt{t^2+1}$

So, $M = \boxed{2\sqrt{2}-1}$