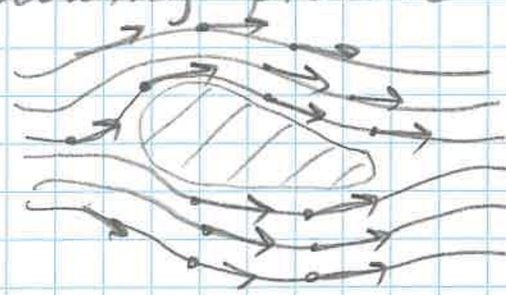
§15.2 Vector Fields and Line Integrals: Work, Circulation, Flux.

- Picture a region in the plane or space occupied by a moving fluid (air, water). → made of particles.
- $\vec{v}(t)$ is the velocity of a particle at time t. Think of vector $\vec{v}(t)$ attached to the pt.
- Velocities $\vec{v}(t)$ may be different at different times. Taking a "snapshot" of velocities vectors at some pts at a particular time can result in the following picture:



Velocity vectors of a flow around an airfoil in a wind tunnel

example of vector field.

Def: A vector field on a domain in the plane or in space is a function that assigns a vector to each pt in this domain.

3D: $\vec{F}(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$

$\vec{F}$ is continuous if $\underbrace{M, N, P}_{\text{component functions}}$ are continuous

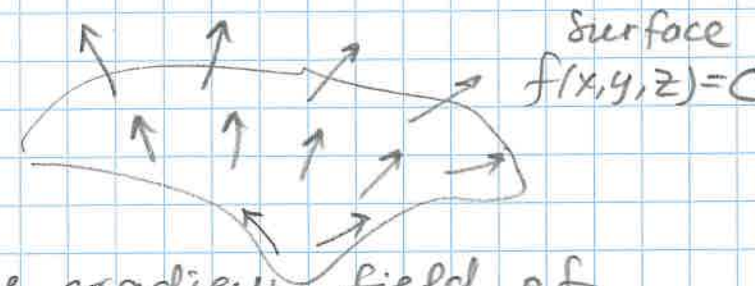
(Pictures on pp. 828-830)

• Gradient Fields

Given $f(x,y,z)$ (differentiable), the gradient field is the field of gradient vectors $\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$

Recall: at any pt. (x,y,z), ∇f points in the direction of greatest increase of f, given by $|\nabla f|$.

Gradient field:



Example 1: Find the gradient field of $g(x,y,z) = xy + yz + xz$

Sol: $\nabla g = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k}$
(e.g., at pt. $(1,2,-1)$, $\nabla g = \langle 1,0,3 \rangle$)

• Work Done by a Force over a Curve in 3D:

- Vector field: $\vec{F} = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$ → represents a force throughout a region in space.

- Let $\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}$, $a \le t \le b$, be a smooth curve in the region.

(Recall: Work $W = \vec{F} \cdot \vec{D}$
force ↗ ↖ displacement)

$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$ $\left(\vec{v} = \frac{d\vec{r}}{dt}\right)$ is the unit tangent vector at a pt. on the curve. Also: $\vec{T} = \frac{d\vec{r}}{ds}$

$\vec{F} \cdot \vec{T}$ will give us work done by a force $\vec{F}$ at a chosen pt on the curve in the direction of motion, i.e., $\vec{T}$. Summing over all pts on the curve results in

$$W = \int_C \vec{F}\cdot\vec{T}\,ds$$, work done by the force $\vec{F}$ in moving the object from point at $t=a$ to pt. at $t=b$, and

$$W = \int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}(\underbrace{\vec{r}(t)}_{\text{parameterization of } C}) \cdot \frac{\vec{v}}{|\vec{v}|}\,ds$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \underbrace{\frac{d\vec{r}/dt}{|\vec{v}|}}_{T} \underbrace{|\vec{v}|\,dt}_{ds} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\,dt$$

also

$$= \int_a^b \Big(M \underbrace{g'(t)}_{\frac{dx}{dt}} + N \underbrace{h'(t)}_{\frac{dy}{dt}} + P \underbrace{k'(t)}_{\frac{dz}{dt}}\Big)\,dt \overset{\text{or}}{=} \int_a^b M\,dx + N\,dy + P\,dz$$

Example 1: Find the work done by force field $\vec{F} = (2y)\vec{i} + (3x)\vec{j} + (x+y)\vec{k}$ along the curve $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + \left(\frac{t}{6}\right)\vec{k}$, $0 \le t \le 2\pi$.

Solution:

helix

1) Evaluate $\vec{F}$ on $\vec{r}(t)$:

$$\vec{F}(\vec{r}(t)) = (2\sin t)\vec{i} + (3\cos t)\vec{j} + (\cos t + \sin t)\vec{k}$$

2) Find $\dfrac{d\vec{r}}{dt} = (-\sin t)\vec{i} + (\cos t)\vec{j} + \frac{1}{6}\vec{k}$

3) Find $\vec{F}(\vec{r}(t)) \cdot \dfrac{d\vec{r}}{dt} = -2\sin^2 t + 3\cos^2 t + \dfrac{\cos t + \sin t}{6}$

So, work $W = \displaystyle\int_0^{2\pi} \left(-2\sin^2 t + 3\cos^2 t + \frac{1}{6}\cos t + \frac{1}{6}\sin t\right) dt$

$$= \int_0^{2\pi} \left(-\frac{2}{2}(1-\cos 2t) + \frac{3}{2}(1+\cos 2t) + \frac{1}{6}\cos t + \frac{1}{6}\sin t \right) dt$$

$$= \left[-\left(t - \frac{1}{2}\sin 2t\right) + \frac{3}{2}\left(t + \frac{1}{2}\sin 2t\right) + \frac{1}{6}\sin t - \frac{1}{6}\cos t \right]_0^{2\pi}$$

$$= \boxed{\pi}$$

(See Examples 5, 6, pages 833–835)

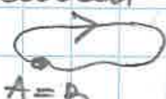
- Flow Integrals & Circulation for Velocity Fields

Let $\vec{F}$ represent the velocity field of a fluid flowing through a region in space. In this context, $\int_C \vec{F}\cdot\vec{T}\,ds$ gives the fluid's flow along or circulation around the curve.

Def: If $\vec{r}(t)$ parameterizes a smooth curve C in the domain of continuous velocity field $\vec{F}$, then

$$\text{Flow along } C \text{ from } A=\vec{r}(a) \text{ to } B=\vec{r}(b) = \underbrace{\int_C \vec{F}\cdot\vec{T}\,ds}_{\text{flow integral}}$$

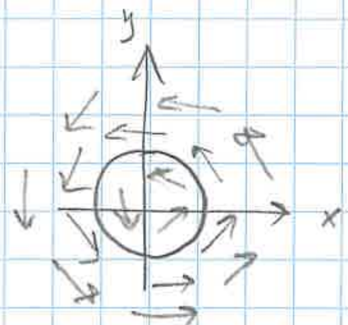
If $A = B \Rightarrow$ flow is called the circulation around the curve.



Direction matters! If we reverse the direction, $\vec{T}$ is replaced by $-\vec{T}$ and the sign of integral changes.

(Similar to work integrals.)

Example 2: Find the circulation of the field $\vec{F} = (x-y)\vec{i} + x\vec{j}$ around the circle $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j}$, $0 \le t \le 2\pi$



Sol: on the circle $\vec{r}(t)$,

$$\vec{F}(\vec{r}(t)) = (\cos t - \sin t)\vec{i} + (\cos t)\vec{j}$$

$$\frac{d\vec{r}}{dt} = (-\sin t)\vec{i} + (\cos t)\vec{j}$$

Then $\vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} = -\sin t\cos t + \underbrace{\sin^2 t + \cos^2 t}_{1}$

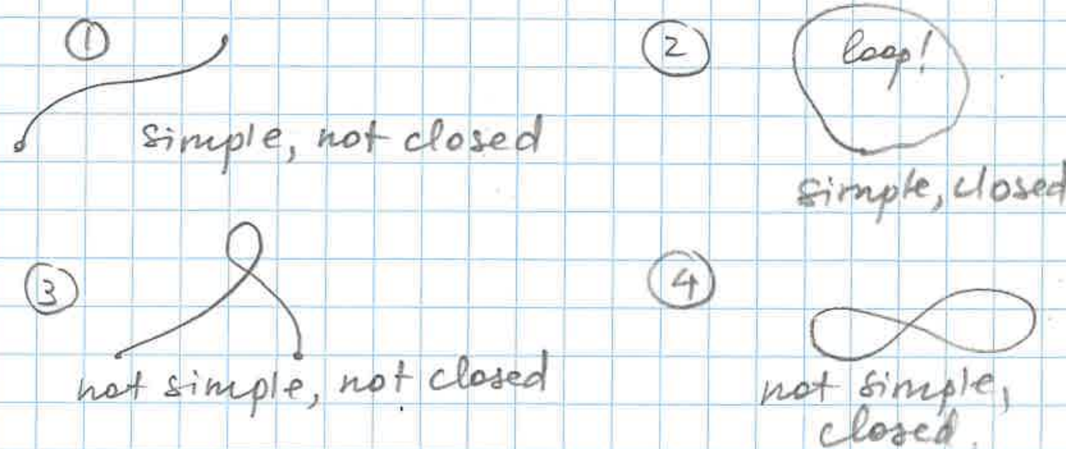
$\Rightarrow$ Circulation $= \int_0^{2\pi} (1 - \sin t \cos t)\, dt = \boxed{2\pi} > 0$

↓ positive!

(counterclockwise direction)

• Flux Across a Simple Closed Plane Curve

Curves:



• Flux (Latin for "flow") gives the rate at which a fluid enters or leaves a region enclosed by a smooth closed plane curve C.

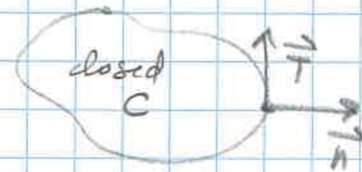
$$\text{Flux of } \underset{\text{(field)}}{\vec{F}} \text{ across } \underset{\text{(curve)}}{C} = \int_C \overbrace{\vec{F} \cdot \vec{n}}^{\text{normal component of } \vec{F}}\, ds$$

"source" of flux

<u>Note</u>: we use normal component of $\vec{F}$, <u>not</u> tangent as in circulation $(\vec{F}\cdot\vec{T})$:

$\vec{F}\cdot\vec{n}$ is the scalar component of $\vec{F}$ in the direction of the curve's outward-pointing normal vector $\vec{n}$.

Q: How to evaluate $\int_C \vec{F}\cdot\vec{n}\,ds$

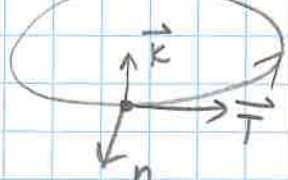


If $\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$, then

$$\text{Flux of } \vec{F} \text{ across } C = \oint_C \underbrace{M\,dy - N\,dx}_{} = \int_a^b \left(M\frac{dy}{dt} - N\frac{dx}{dt}\right)dt$$

(integral around C in the counterclockwise dir.)

compact form: do not need $\vec{n}$ or ds! (based on $\vec{n} = \vec{T}\times\vec{k}$)



$\vec{k} = \langle 0,0,1\rangle$ 3rd standard unit vector

$$\vec{n} = \vec{T}\times\vec{k} = \left(\frac{dx}{ds}\,\vec{i} + \frac{dy}{ds}\,\vec{j}\right)\times\vec{k} = \frac{dy}{ds}\,\vec{i} - \frac{dx}{ds}\,\vec{j}$$

$$\Rightarrow \vec{F}\cdot\vec{n} = M\frac{dy}{ds} - N\frac{dx}{ds}$$

If motion is clockwise:  $\vec{k}\times\vec{T}$ points outward

counterclockwise  $\vec{T}\times\vec{k}$ points outward

usual choise is $\vec{n} = \vec{T}\times\vec{k}$ $\Rightarrow$

in $\oint_C$ we assume a counterclockwise direction.

Flux can be positive (flow leaves the closed area $\longleftrightarrow$ net flow across C is outward)

Flux can be negative (flow enters the area $\longleftrightarrow$ net flow across C is inward)

Example 3: Find the flux of $\vec{F} = x\vec{i} + y\vec{j}$ across the circle $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j}$, $0 \le t \le 2\pi$

Solution:

$$\text{Flux of } \vec{F} \text{ across } C = \oint_C \underset{\text{on } \vec{r}(t)}{M\,dy - N\,dx}$$

$$= \int_0^{2\pi} \left[(\cos t)(\cos t) - (\sin t)(-\sin t)\right] dt = \int_0^{2\pi} 1\, dt = \boxed{2\pi}$$

flux > 0 $\to$ the net flow across C is outward.