§ 15.3 Path Independence, Conservative Fields, Potential Functions.

Def: Let $\vec{F}$ be a vector field defined on an open region D in space, and suppose that for any two pts A & B in D,

$\int_C \vec{F}\cdot d\vec{r}$ is the same over all paths from A to B. Then $\int_C \vec{F}\cdot d\vec{r}$ is path independent in D & the field $\vec{F}$ is conservative on D.

$$\int_C \vec{F}\cdot d\vec{r} = \int_A^B \vec{F}\cdot d\vec{r}$$

(work integral or flow integral)

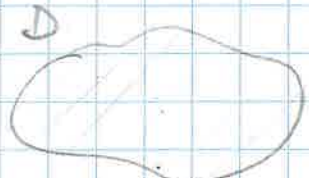
If $\vec{F} = \nabla f$ for some scalar func. f on D, then f is called a potential function for $\vec{F}$.

$$\underbrace{\int_A^B \vec{F}\cdot d\vec{r} = \int_A^B \nabla f\cdot d\vec{r} = f(B) - f(A)}_{\text{FTC}}$$

(see next page)

Some assumptions on D:
- curves are piecewise smooth
- domain D is connected (any 2 pts can be joined by a curve lying in D)

D



simply connected ⟶ every loop can be contracted to a pt. in D without leaving D.

D

Not simply connected

<u>Theorem 1</u> Let C be a smooth curve joining A to B in the plane or space, parameterized by $\vec{r}(t)$. Let f be a differentiable func. with continuous gradient $\vec{F} = \nabla f$ on D containing C.

Then $\displaystyle\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A)$

(<u>Fund. thm</u> of <u>line</u> <u>integrals</u>)

<u>Example 1</u> : Work done by a conservative field

$$\vec{F} = yz\vec{i} + xz\vec{j} + xy\vec{k} = \nabla(\underbrace{xyz})$$

potential func. $f = xyz$

Any smooth curve C joining $A(-1,3,9)$ to $B(1,6,-4)$

$$\int_A^B \vec{F}\cdot d\vec{r} = \int_A^B \nabla f\cdot d\vec{r} = f(B) - f(A)$$

$$= xyz\Big|_{(1,6,-4)} - xyz\Big|_{(-1,3,9)} = (1)(6)(-4) - (-1)(3)(9)$$

$$= \boxed{3}$$

<u>Theorem 2.</u> Conservative fields are gradient fields

Let $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ be a vector field w/ continuous components M, N, P on a region D. Then $\vec{F}$ is conservative $\Leftrightarrow$ $\vec{F} = \nabla f$ for a diff. func. f.

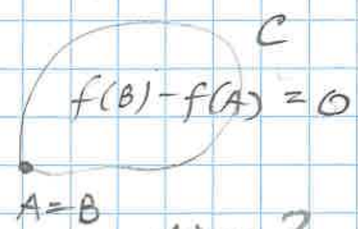
(That is, $\vec{F} = \nabla f$ $\Leftrightarrow$ for any A & B,

$\displaystyle\int_A^B \vec{F}\cdot d\vec{r}$ is the same for any C joining A & B.)

Theorem 3. The following statements are equivalent:

(1) $\oint_C \vec{F} \cdot d\vec{r} = 0$ around every loop C in D

(2) $\vec{F}$ is conservative on D.



Q's: How do we know if $\vec{F}$ is conservative?
If so, how do we find f s.t. $\vec{F} = \nabla f$?

- Component Test for Conservative Fields

$\vec{F} = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$ - field on a simply connected domain D.

$\vec{F}$ is conservative $\Leftrightarrow \dfrac{\partial P}{\partial y} = \dfrac{\partial N}{\partial z}, \dfrac{\partial M}{\partial z} = \dfrac{\partial P}{\partial x}, \dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$

Example 2: Determine whether

$\vec{F} = \underbrace{e^x \cos y}_{M} \vec{i} - \underbrace{e^x \sin y}_{N} \vec{j} + \underbrace{z}_{P} \vec{k}$ is conservative.

$\dfrac{\partial P}{\partial y} = 0 = \dfrac{\partial N}{\partial z}$; $\quad \dfrac{\partial M}{\partial z} = 0 = \dfrac{\partial P}{\partial x}$;

$\dfrac{\partial N}{\partial x} = -e^x \sin y = \dfrac{\partial M}{\partial y} \quad \Rightarrow$ Yes, conservative!

- How do we find f s.t. $\vec{F} = \nabla f$? Integrate equations:

$\underbrace{\dfrac{\partial f}{\partial x} = \underbrace{e^x \cos y}_{M}}_{(1)}, \quad \underbrace{\dfrac{\partial f}{\partial y} = \underbrace{-e^x \sin y}_{N}}_{(2)}, \quad \underbrace{\dfrac{\partial f}{\partial z} = \underbrace{z}_{M}}_{(3)}$

From (1): $f(x,y,z) = \int e^x \cos y \, dx = e^x \cos y + \underbrace{g(y,z)}_{\text{"constant", depends on } y,z}$

$$\Rightarrow \frac{\partial f}{\partial y} = -e^x \sin y + \frac{\partial g}{\partial y} = -e^x \sin y \Rightarrow \frac{\partial g}{\partial y} = 0$$

$\Rightarrow \quad g = g(\underline{\underline{z}})$ only, so

$$f(x,y,z) = e^x \cos y + g(z)$$

$$\Rightarrow \frac{\partial f}{\partial z} = \frac{dg}{dz} = z \Rightarrow g(z) = \frac{z^2}{2} + C$$

Hence, $\boxed{f(x,y,z) = e^x \cos y + \frac{z^2}{2} + C}$

(Check: $\frac{\partial f}{\partial x} = e^x \cos y$, $\frac{\partial f}{\partial y} = -e^x \sin y$, $\frac{\partial f}{\partial z} = z$)

- <u>Exact Differential Forms:</u>

$$\underbrace{M\,dx + N\,dy + P\,dz}_{\text{differential form.}}$$

It is exact if $M\,dx + N\,dy + P\,dz =$

$= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = df$ for some func. f.

- $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ is conservative <u>if and only if</u> $M\,dx + N\,dy + P\,dz$ is exact which is true <u>if & only if</u> $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$