§ 15.4 Green's Theorem in the Plane.

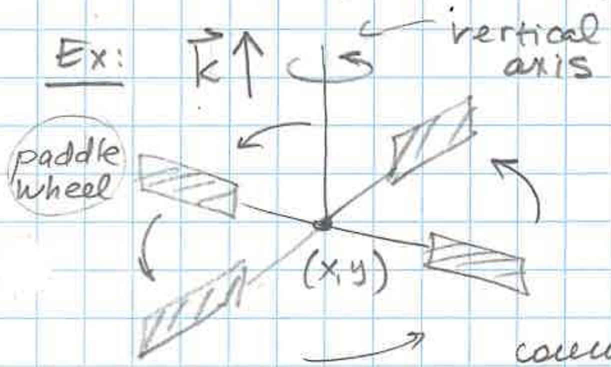
Consider $\vec{F} = M(x,y)\vec{i} + N(x,y)\vec{j}$ - a velocity field of a fluid flowing in the plane.

Def: The $\boxed{\text{circulation density}}$ of $\vec{F}$

at the pt. (x,y) is the scalar expression
$$\underbrace{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}$$

Also called the $\vec{k}$-component of the curl, denoted by $(\text{curl}\,\vec{F})\cdot\vec{k}$. It measures the rate of the fluid's rotation at a pt. in the plane

Ex:



$\text{Curl}\,\vec{F}\cdot\vec{k} > 0$ at (x,y) (<0 for clockwise)

counterclockwise rotation

(Curl $\vec{F}$ is a more general circulation vector field we'll talk about in § 15.7; here we need its $\vec{k}$-component only.
Curl $\vec{F}$ is circulation divided by area.)

- $\boxed{\text{Divergence}}$: (or $\boxed{\text{flux density}}$) of $\vec{F} = M\vec{i} + N\vec{j}$

at (x,y) is $\text{div}\,\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$ (scalar!).
$$\left(\text{"}\nabla\cdot\vec{F}\text{"} = \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}\rangle \cdot \langle M, N\rangle\right)$$

- Think of a compessible gas:

expanding at (x,y) (lines diverge from (x,y))

$\Rightarrow \text{div}\,\vec{F} > 0$

compressed at (x,y)

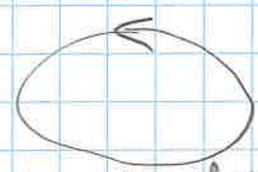
$\Rightarrow \text{div}\,\vec{F} < 0$

Example 1: Find $\operatorname{div}\vec{F}$ of $\vec{F} = \underbrace{(x^2-y)}_{M}\vec{i} + \underbrace{(xy-y^2)}_{N}\vec{j}$

Sol: $\operatorname{div}\vec{F} = 2x + (x-2y) = \boxed{3x-2y}$

$\operatorname{div}\vec{F}(2,0) = 6 > 0 \rightarrow$ expansion
$\operatorname{div}\vec{F}(0,2) = -4 < 0 \longrightarrow$ compession
$\operatorname{div}F(0,0) = 0 \longrightarrow$ neither!

• Two Forms of Green's Theorem.

C 

counterclockwise: positively oriented

C 

clockwise: negatively oriented

Turn line integrals into double integrals

① Green's Thm: Circulation-Curl / Tangential Form.

Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\vec{F} = M\vec{i} + N\vec{j}$ be a vector field with M & N having continuous first partial derivatives on R. Then the counterclock circulation of $\vec{F}$ around C is

$$(1) \quad \underbrace{\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + N\,dy}_{\text{Counterclockwise circulation}} = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy}_{\text{Curl integral}}$$

$$= \iint_R (\operatorname{curl}\vec{F})\cdot\vec{k}\,dx\,dy$$

② Green's Thm: Flux Divergence / Normal Form.

Same assumptions as in Thm ①. Then the outward flux of $\vec{F}$ across C is

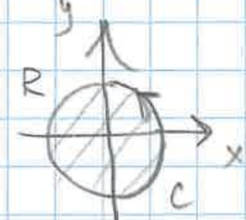
$$(2) \quad \underbrace{\oint_C \vec{F}\cdot\vec{n}\,ds = \oint M\,dy - N\,dx}_{\text{outward flux}} = \underbrace{\iint_R \overbrace{\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)}^{=\operatorname{div}\vec{F}} dx\,dy}_{\text{divergence integral}}$$

Note: the 2 forms of Green's Thm are equivalent $\rightsquigarrow$ apply eqn (1) to the field $\vec{G}_1 = -N\vec{i} + M\vec{j}$ to get (2) or apply eqn (2) to $\vec{G}_2 = N\vec{i} - M\vec{j}$ to get (1).

Example 1: Verify ~~both~~ forms of Green's thm for the vector field $\vec{F}(x,y) = (x-y)\vec{i} + x\vec{j}$ and the region R enclosed by the unit circle C: $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j}$, $0 \le t \le 2\pi$.

Solution: (1) $$\underbrace{\oint_C M\,dx + N\,dy}_{\text{circ.}} \overset{?}{=} \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}_{\text{curl integral}}$$

$M = x - y = \cos t - \sin t$
$N = x = \cos t$
$dx = x'\,dt = (-\sin t)\,dt$
$dy = y'\,dt = (\cos t)\,dt$



$$\frac{\partial N}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = -1$$

So, $$\oint_C M\,dx + N\,dy = \int_0^{2\pi} \left((\cos t - \sin t)(-\sin t) + \cos^2 t\right) dt$$

$$= \int_0^{2\pi} (1 - \cos t \sin t)\,dt = \boxed{2\pi} \checkmark$$

$$\iint_R (1 - (-1))\,dx\,dy = 2\underbrace{\iint_R dx\,dy}_{\text{area inside the circle is } \pi} = \boxed{2\pi} \checkmark$$

(2) $$\oint_C M\,dy - N\,dx \overset{?}{=} \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$

$$\oint_C M\,dy - N\,dx = \int_0^{2\pi} \left[(\cos t - \sin t)(\cos t) + \cos t \sin t\right] dt$$

$$= \int_0^{2\pi} \cos^2 t\,dt = \boxed{\pi} \checkmark$$

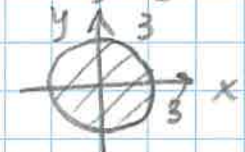
$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R (1+0)\,dx\,dy$$

$$\left(\frac{\partial M}{\partial x} = 1\,,\ \frac{\partial N}{\partial y} = 0\right) \qquad = \underbrace{\iint_R dx\,dy}_{\text{area inside circle}} = \boxed{\pi}\ \checkmark$$

Example 2.

Evaluate $\displaystyle\oint_C \overbrace{\left(3y - e^{\sin x}\right)dx + \left(7x + \sqrt{y^4+1}\right)dy}^{\text{Gross!}}$

where C is the circle $x^2+y^2=9$.



Solution: use Green's thm ①

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

with $M = 3y - e^{\sin x}$
$N = 7x + \sqrt{y^4+1}$

$$\oint_C M\,dx + N\,dy = \iint_R \overbrace{(7-3)}^{\text{easy!}}\,dx\,dy$$

$$= \underbrace{\int_0^{2\pi}\int_0^3}_{\text{polar coord's}} 4\underbrace{r\,dr\,d\theta}_{dA} = \int_0^{2\pi}\left[\frac{4r^2}{2}\right]_0^3 d\theta = \int_0^{2\pi} 18\,d\theta = \boxed{36\pi}$$

① Can use form ② as well:

$$\oint_C M\,dy - N\,dx = \iint_R \overbrace{\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)}^{\text{easy!}} dx\,dy$$

with $M = 7x + \sqrt{y^4+1}$ & $N = -3y + e^{\sin x}$

(Try it!)

(See also Exercises 4, 5 on page 859)