§15.5 Surfaces & Area

- Curves:
  - $y = f(x)$ explicit form
  - $F(x,y) = 0$ implicit form
  - $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ parametric form

    $a \le t \le b$

- Surfaces:
  - $z = f(x,y)$ explicit form
  - $F(x,y,z) = 0$ implicit form
  - parametric form ?

- Parameterization of surfaces:

$$\underbrace{\vec{r}(u,v) = f(u,v)\vec{i} + g(u,v)\vec{j} + h(u,v)\vec{k}}$$

continuous vector func. defined on a region R in uv-plane.

u, v parameters (R is domain for u & v)

Range of $\vec{r}$ is the surface traced by $\vec{r}$

($\vec{r}$ needs to be one-to-one on R to make sure the surface does not cross itself!)

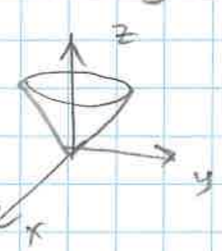
$\vec{r}(u,v)$ is equivalent to set of parametric eqn's

$$\begin{cases} x = f(u,v) \\ y = g(u,v) \\ z = h(u,v) \end{cases}$$

Example 1: Parameterization of the cone

$$z = \sqrt{x^2+y^2}, \quad z \in [0,1]$$

(a) Use cylindrical coord's:



$z = \sqrt{x^2+y^2} = r$, $0 \le r \le 1$ and $0 \le \theta \le 2\pi$

$x = r\cos\theta$, $y = r\sin\theta$. Thus,

$$\vec{r}(r,\theta) = (r\cos\theta)\vec{i} + (r\sin\theta)\vec{j} + (r)\vec{k}.$$

(b) "Stupid" parameterization:

$$\vec{r}(x,y) = x\vec{i} + y\vec{j} + \sqrt{x^2+y^2}\,\vec{k}, \quad x,y \in \mathbb{R}$$

(c) Spherical coord's? Try:

$$\underbrace{x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi}_{\text{too many var's!?}}$$

Note $z = \sqrt{x^2+y^2} \Rightarrow \varphi = \frac{\pi}{4}$ always $\Rightarrow$ $(\sin\varphi = \sqrt{2}/2)$

So, $\vec{r}(\rho,\theta) = \left(\frac{\sqrt{2}}{2}\rho\cos\theta\right)\vec{i} + \left(\frac{\sqrt{2}}{2}\rho\sin\theta\right)\vec{j} + \left(\frac{\sqrt{2}}{2}\rho\right)\vec{k}$

$$0 \le \rho \le \sqrt{2}, \quad 0 \le \theta \le 2\pi$$

$\Uparrow$

$0 \le z \le 1$ (given)

and $0 \le \rho\frac{\sqrt{2}}{2} \le 1$

Look at Examples 2, 3 on pages 864-865.

- Surface Area

$\vec{r}(u,v) = f(u,v)\vec{i} + g(u,v)\vec{j} + h(u,v)\vec{k}, \quad a \le u \le b, \ c \le v \le d$

Consider $\vec{r}_u = \frac{\partial \vec{r}}{\partial u} = \frac{\partial f}{\partial u}\vec{i} + \frac{\partial g}{\partial u}\vec{j} + \frac{\partial h}{\partial u}\vec{k}$

& $\vec{r}_v = \frac{\partial \vec{r}}{\partial v} = \frac{\partial f}{\partial v}\vec{i} + \frac{\partial g}{\partial v}\vec{j} + \frac{\partial h}{\partial v}\vec{k}$

Def: $\vec{r}(u,v)$ is smooth if $\vec{r}_u$ & $\vec{r}_v$ are continuous & $\vec{r}_u \times \vec{r}_v \ne \vec{0}$ on the parameter domain interior.

Def: The area of the smooth surface

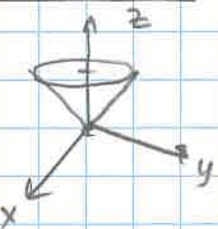
$S: \vec{r}(u,v) = f(u,v)\vec{i} + g(u,v)\vec{j} + h(u,v)\vec{k}$, $a \le u \le b$, $c \le v \le d$, is given by

$$\text{Area} = \iint_R |\vec{r}_u \times \vec{r}_v| \underbrace{dA}_{du\,dv} = \int_c^d\int_a^b \underbrace{|\vec{r_u} \times \vec{r_v}|\, du\, dv}_{d\sigma}$$

$$= \iint_S d\sigma$$

surface area differential

Example 2: Find the surface area of the cone $z = \sqrt{x^2+y^2}$, $0 \le z \le 1$



We already know the parametric eqn in cylindrical coord's:

$$\vec{r}(r,\theta) = (r\cos\theta)\vec{i} + (r\sin\theta)\vec{j} + r\vec{k}, \quad 0 \le r \le 1, \; 0 \le \theta \le 2\pi.$$

$$\vec{r}_r = (\cos\theta)\vec{i} + (\sin\theta)\vec{j} + \vec{k}, \quad \vec{r}_\theta = (-r\sin\theta)\vec{i} + (r\cos\theta)\vec{j} + 0\vec{k}$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-r\cos\theta)\vec{i} - (r\sin\theta)\vec{j} + (\overbrace{r\cos^2\theta + r\sin^2\theta}^{r})\vec{k}$$

$$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2r^2} = \sqrt{2}\, r$$

$$\Rightarrow A = \int_0^{2\pi}\int_0^1 |\vec{r}_r \times \vec{r}_\theta|\, dr\, d\theta = \int_0^{2\pi}\int_0^1 \sqrt{2}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{\sqrt{2}}{2} r^2\right]_0^1 d\theta = \frac{\sqrt{2}}{2} \cdot 2\pi = \boxed{\pi\sqrt{2}} \text{ (square units)}$$

→ Read Examples 5, 6, pp. 867-869

→ Ignore "Implicit Surfaces".

Example 2 → revisited:

What if the cone is parameterized by

$$\vec{r}(x,y) = x\vec{i} + y\vec{j} + \sqrt{x^2+y^2}\,\vec{k}\,?$$

The result will be the same:

$$\vec{r}_x = 1\vec{i} + 0\vec{j} + \frac{x}{\sqrt{x^2+y^2}}\vec{k}$$

$$\vec{r}_y = 0\vec{i} + 1\vec{j} + \frac{y}{\sqrt{x^2+y^2}}\vec{k}$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & x/\sqrt{x^2+y^2} \\ 0 & 1 & y/\sqrt{x^2+y^2} \end{vmatrix} =$$

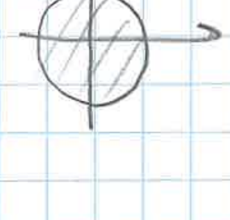
$$= \left(-\frac{x}{\sqrt{x^2+y^2}}\right)\vec{i} - \left(\frac{y}{\sqrt{x^2+y^2}}\right)\vec{j} + 1\vec{k}$$

$$\Rightarrow |\vec{r}_x \times \vec{r}_y| = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1} = \sqrt{2}$$

$$\Rightarrow A = \iint_R \sqrt{2}\,dx\,dy = \int_0^{2\pi}\int_0^1 \sqrt{2}\,r\,dr\,d\theta = \boxed{\pi\sqrt{2}}$$

(as before)

R



$x^2+y^2=1$ $(0 \le z \le 1)$