§ 15.6 Surface Integrals.

smooth $\to$ $\vec{r}_u, \vec{r}_v$ are cont., $\vec{r}_u \times \vec{r}_v \neq \vec{0}$

Given a surface S parameterized by

$$\vec{r}(u,v) = f(u,v)\vec{i} + g(u,v)\vec{j} + h(u,v)\vec{k}, \text{ with}$$

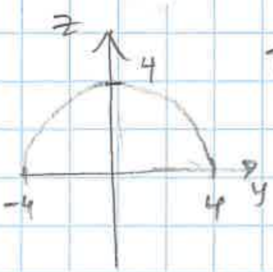
$(u,v) \in R$ (region),

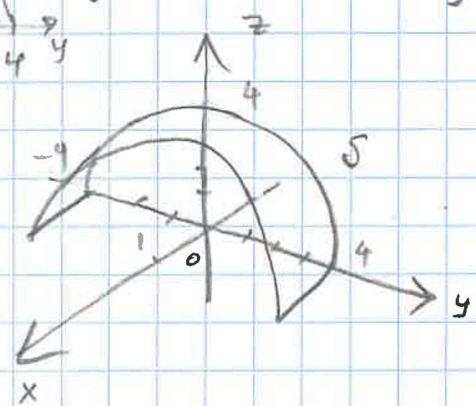
The surface integral of a func. $G(x,y,z)$ (continuous) defined on S, is given by:

$$\underbrace{\iint_S}_{\text{surface}} G(x,y,z)\, d\sigma = \underbrace{\iint_R}_{(u,v)\in R} \underbrace{G(f(u,v), g(u,v), h(u,v))}_{G \text{ on } \vec{r}(u,v)} \underbrace{|\vec{r}_u \times \vec{r}_v|\, du\, dv}_{d\sigma}$$

($G = 1 \Rightarrow$ surface area)

Example 1: Integrate $g(x,y,z) = x\sqrt{y^2+4}$ over the surface cut from the parabolic cylinder $y^2 + 4z = 16$ by the planes $x=0$, $x=1$, & $z=0$.





$$z = \frac{16-y^2}{4} = 4 - \frac{1}{4}y^2$$

$$S: \vec{r}(x,y) = x\vec{i} + y\vec{j} + \left(4 - \frac{1}{4}y^2\right)\vec{k}$$

$$0 \le x \le 1, \quad -4 \le y \le 4$$

$$\vec{r}_x = 1\vec{i} + 0\vec{j} + 0\vec{k}$$

$$\vec{r}_y = 0\vec{i} + 1\vec{j} - \frac{y}{2}\vec{k}$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & -\frac{y}{2} \end{vmatrix} = 0\vec{i} + \frac{1}{2}y\vec{j} + 1\vec{k}$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{\frac{1}{4}y^2 + 1}$$

Thus, $\iint_S g \, d\sigma =$

$$= \int_{-4}^{4}\int_{0}^{1} \left(x\sqrt{y^2+4}\right)\Big(\overbrace{\sqrt{\tfrac{1}{4}y^2+1}}^{=\sqrt{y^2+4}/2}\Big)\,dx\,dy$$

$$= \int_{-4}^{4}\int_{0}^{1} \frac{x}{2}\left(y^2+4\right)\,dx\,dy = \int_{-4}^{4}\left[\frac{x^2}{4}\left(y^2+4\right)\right]_0^1 dy$$

$$= \int_{-4}^{4} \frac{y^2+4}{4}\,dy = \left[\frac{y^3}{12}+y\right]_{-4}^{4} = \boxed{56/3}$$

Example 2: Integrate $f(x,y,z) = x^2$ over the cone $z = \sqrt{x^2+y^2}$, $0 \le z \le 1$.

Recall: $\vec{r}(r,\theta) = (r\cos\theta)\vec{i} + (r\sin\theta)\vec{j} + r\vec{k}$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$

& $|\vec{r}_r \times \vec{r}_\theta| = \sqrt{2}\,r$

Then $$\iint_S f\,d\sigma = \int_0^{2\pi}\int_0^1 \underbrace{(r\cos\theta)^2}_{x^2 \text{ on } S}\ \underbrace{\sqrt{2}\cdot r\,dr\,d\theta}_{d\sigma}$$

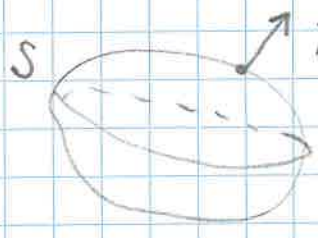
$$= \sqrt{2}\int_0^{2\pi}\int_0^1 r^3\cos^2\theta\,dr\,d\theta = \frac{\sqrt{2}}{4}\int_0^{2\pi}\cos^2\theta\,d\theta$$

$$= \frac{\sqrt{2}}{4}\left[\frac{\theta}{2}+\frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \boxed{\frac{\pi\sqrt{2}}{4}}$$

→ Read Examples 2-4, pp. 876-878

• Orientation of a Surface:

Given a surface S, its parameterization $\vec{r}(u,v)$ gives a vector $\vec{r}_u \times \vec{r}_v$ that is normal to the surface. If S has 2 sides, $-\vec{r}_u \times \vec{r}_v$



is also normal to S. We can give S an orientation by choosing a normal vector $\vec{n}$.

A smooth surface S is orientable (or two-sided) if it is possible to define a field $\vec{n}$ of unit normal vectors on S which varies continuously w/ position. Given field $\vec{n}$, S is oriented in space. By convention, we usually choose $\vec{n}$ on a closed surface to point outward.

Ex. of a not orientable surface:

Möbius band (one-sided!) (Fig. 15.50, p. 878)

Def. The Surface Integral for Flux:

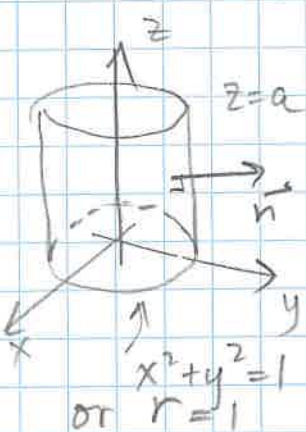
if $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ is a field w/ continuous M, N, P over a smooth oriented surface S, then the flux of $\vec{F}$ across S is

$$\iint_S \vec{F} \cdot \underbrace{\vec{n}}_{\text{direction}} \, d\sigma$$ (surface integral of $\vec{F}$ over S)

If $\underset{(\text{unit})}{\vec{n}} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ and $d\sigma = |\vec{r}_u \times \vec{r}_v| \, du dv$ $\vec{r}(u,v)$

then $$\iint_S \vec{F} \cdot \vec{n} \, d\sigma = \boxed{\iint_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \, du dv}$$

Example 3: Find the flux $\iint_S \vec{F}\cdot\vec{n}\, d\sigma$ across the cylinder $x^2+y^2=1$ cut by the planes $z=0$ and $z=a$ $(a>0)$ if $\vec{F}=x\vec{i}+y\vec{j}+z\vec{k}$, in outward direction.



Let us parameterize the surface in cylindrical coord's

$$x = r\cos\theta = \cos\theta \quad (r=1)$$
$$y = r\sin\theta = \sin\theta$$
$$z = z$$
$$0 \le \theta \le 2\pi, \qquad 0 \le z \le a$$

$$\Rightarrow \vec{r}(\theta,z) = (\cos\theta)\vec{i} + (\sin\theta)\vec{j} + z\vec{k}, \quad 0\le z\le a, \quad 0\le\theta\le 2\pi$$

$$\vec{r}_\theta = (-\sin\theta)\vec{i} + (\cos\theta)\vec{j} + 0\vec{k}$$
$$\vec{r}_z = 0\vec{i} + 0\vec{j} + 1\vec{k}$$

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta)\vec{i} + (\sin\theta)\vec{j} + 0\vec{k}$$

$$\Rightarrow \iint_S \underbrace{\left((\cos\theta)\vec{i} + (\sin\theta)\vec{j} + z\vec{k}\right)}_{\vec{F} \text{ in terms of } \theta \,\&\, z} \cdot \underbrace{\overbrace{\left((\cos\theta)\vec{i} + (\sin\theta)\vec{j}\right)}^{\vec{r}_\theta\times\vec{r}_z} dz\, d\theta}_{d\sigma}$$

$$= \iint_S (1)\, dz\, d\theta = \int_0^{2\pi}\int_0^a (1)\, dz\, d\theta = \int_0^{2\pi} a\, d\theta = \boxed{2\pi a}$$

flux across S