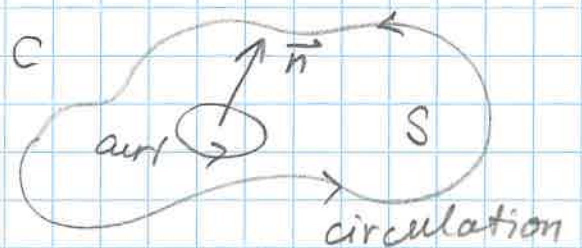
§15.7 Stokes' Theorem

Extends Green's Theorem (Circulation/Curl form) to general surfaces w/ smooth boundaries

If a vector field $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$, then the curl vector is $\text{curl}\,\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\vec{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$



$\longrightarrow$ **Stokes' Theorem:**

S – piecewise smooth oriented surface w/ smooth boundary C (curve)

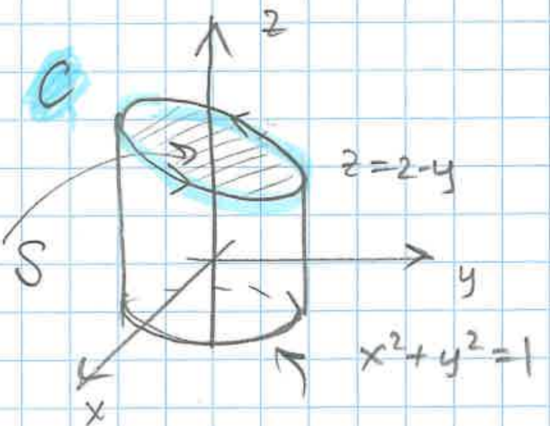
Let $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ be a vector field w/ continuous M, N, P on S. Then

$$\boxed{\underbrace{\oint_C \vec{F} \cdot d\vec{r}}_{\substack{\text{circulation of } \vec{F} \\ \text{around } C \text{ w/} \\ \text{counterclockwise} \\ \text{direction w.r.t. } \vec{n}}} = \underbrace{\iint_S \underbrace{(\nabla \times \vec{F})}_{\text{curl}\,\vec{F}} \cdot \underbrace{\vec{n}}_{\text{normal}}\, d\sigma}_{\substack{\text{curl integral over } S \\ \text{also: flux of curl}\,\vec{F}}}}$$

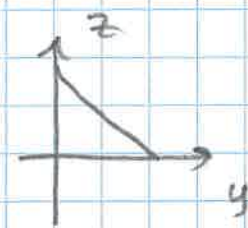
Example: Evaluate $\underbrace{\oint_C \vec{F} \cdot d\vec{r}}_{\text{circ. of } \vec{F} \text{ around } C}$, where $\vec{F} = (-y^2)\vec{i} + x\vec{j} + z^2\vec{k}$

and C is the curve of intersection of the plane $y + z = 2$ & the cylinder $x^2 + y^2 = 1$. Orient C to be counterclockwise when viewed from above.

<u>Solution</u>: Apply Stokes's Thm to evaluate $\oint_C \vec{F}\cdot d\vec{r}$.



$z = 2-y$



C bounds surface S parameterized by

$$\vec{r}(x,y) = x\vec{i} + y\vec{j} + (\underbrace{2-y}_{z})\vec{k}$$

$$-1 \le x \le 1$$
$$-1 \le y \le 1$$

Need: $\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & x & z^2 \end{vmatrix} =$

$M = -y^2$
$N = x$
$P = z^2$

$$= (0-0)\vec{i} - (0-0)\vec{j} + (1+2y)\vec{k}$$

$$= \langle 0, 0, 1-2y \rangle$$

$\vec{n} = \dfrac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|}$: $\vec{r}_x = \langle 1, 0, 0 \rangle$, $\vec{r}_y = \langle 0, 1, -1 \rangle$

$$\Rightarrow \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \langle 0, 1, 1 \rangle$$

$\Downarrow$

$\vec{n} = \dfrac{\langle 0,1,1 \rangle}{\sqrt{2}}$

$\Rightarrow$ Circ. $\oint_C \vec{F}\cdot dr = \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\, d\sigma =$

$$= \iint_S \langle 0,0,1+2y\rangle \cdot \frac{\langle 0,1,1\rangle}{\cancel{\sqrt{2}}}\, \cancel{\sqrt{2}}\, dx\,dy$$

$$= \iint_S (1+2y)\,dx\,dy \underset{PC}{=} \int_0^{2\pi}\int_0^1 \Big(1+2\underbrace{r\sin\theta}_{y}\Big)\underbrace{r\,dr\,d\theta}_{\substack{dx\,dy\\ \text{in PC}}}$$

$$= \int_0^{2\pi} \left(\frac{r^2}{2}+\frac{2r^3}{3}\sin\theta\right)\Big]_0^1 d\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{2}+\frac{2}{3}\sin\theta\right)d\theta = \left[\frac{\theta}{2}-\frac{2}{3}\cos\theta\right]_0^{2\pi}$$

$$= \left(\frac{2\pi}{2}-\frac{2}{3}\cos 2\pi\right)-\left(0-\frac{2}{3}\cos 0\right) = \boxed{\pi}$$

<u>Note</u>: If two surfaces S_1 & S_2 have the same boundary C, their curl integrals are equal:

$$\iint_{S_1} (\nabla\times\vec{F})\cdot\vec{n}_1\,d\sigma = \iint_{S_2}(\nabla\times\vec{F})\cdot\vec{n}_2\,d\sigma$$

(So you could choose the other part of the cylinder to be the surface w/ the same boundary C)

<u>Also</u>:

$$\begin{tikzcd}
\vec{F}\text{-conservative on } D \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & F=\nabla f \text{ on } D \arrow[d, Rightarrow] \\
\oint_C \vec{F}\cdot d\vec{r}=0 & \nabla\times\vec{F}=\vec{0} \text{ in } D \arrow[l, Rightarrow]
\end{tikzcd}$$

over any closed C in D