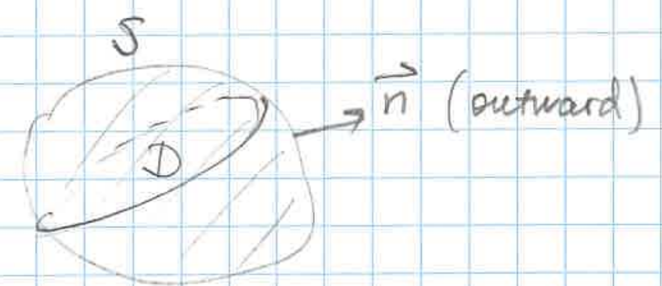
§15.8 The Divergence Theorem.

3D: $\underbrace{\text{div}\,\vec{F}}_{\text{divergence}} = \nabla \cdot \vec{F} = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y} + \dfrac{\partial P}{\partial z}$

Theorem: S: closed, smooth, oriented surface, $\vec{n}$ points outward

$$\boxed{\underbrace{\iint_S \vec{F}\cdot\vec{n}\,d\sigma}_{\text{outward flux of } \vec{F} \text{ across } S} = \underbrace{\iiint_D \underbrace{\nabla\cdot\vec{F}}_{\text{div}\,\vec{F}}\,dV}_{\text{divergence integral}}}$$

$\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$



D - solid enclosed by S

Note: can calculate a flux integral across a surface using triple integral for divergence!

Example: Find the flux of the vector field $F(x,y,z) = z\vec{i} + y\vec{j} + x\vec{k}$ over the unit sphere $x^2+y^2+z^2=1$.

Solution: we need $\text{div}\,\vec{F} = \nabla\cdot\vec{F} =$

$= \dfrac{\partial}{\partial x}(z) + \dfrac{\partial}{\partial y}(y) + \dfrac{\partial}{\partial z}(x) = 0 + 1 + 0 = \boxed{1}$

The Divergence theorem gives:

$$\underbrace{\iint_S \vec{F}\cdot\vec{n}\, d\sigma}_{\text{flux}} = \iiint_D \operatorname{div}\vec{F}\, dV =$$

$$= \underbrace{\iiint_D (1)\, dv}_{\substack{\text{this is just the volume}\\ \text{of the unit ball!}}} = \boxed{\frac{4\pi}{3}} \rightsquigarrow \underline{\text{flux}}$$

$$\left(V = \frac{4}{3}\pi \underbrace{r}_{\text{radius}}^3\right)$$

<u>Q</u>: What if we did not remember the formula for the sphere volume?

- Use spherical coordinates:

$$\iiint_D 1\, dV = \underbrace{\int_0^{2\pi}}_{\theta} \underbrace{\int_0^{\pi}}_{\varphi} \underbrace{\int_0^{1}}_{\rho} \underbrace{\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta}_{dV}$$

$$= \int_0^{2\pi}\int_0^{\pi} \left[\frac{\rho^3}{3}\sin\varphi\right]_0^1 d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{\sin\varphi}{3}\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} \left[-\frac{\cos\varphi}{3}\right]_0^{\pi} d\theta = \frac{1}{3}\int_0^{2\pi} 2\, d\theta = \boxed{\frac{4\pi}{3}}$$

again.

See, it is all about integrals; here to compute $\iint_S \vec{F}\cdot\vec{n}\, d\sigma$ for flux, we used $\iiint_D \operatorname{div}\vec{F}\, dV$. Cool!