

Chapter 11. Homomorphisms

(1)

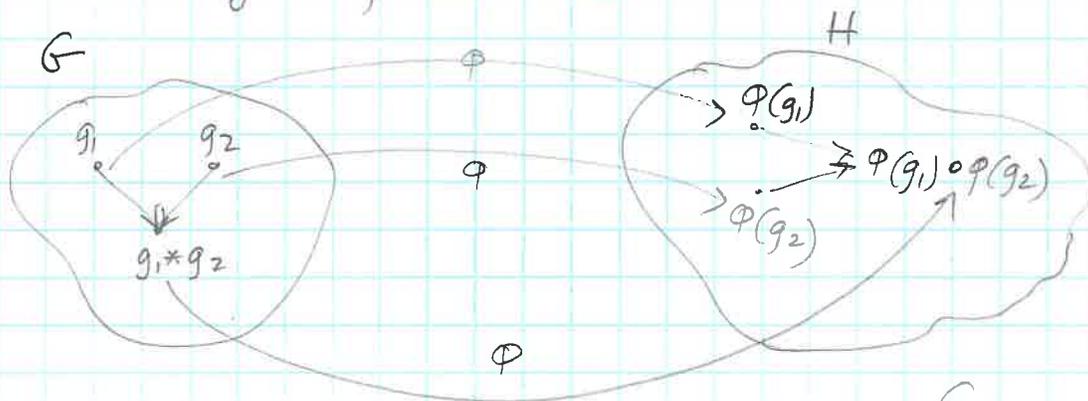
We relax "bijection" required for isomorphism of groups \Rightarrow we get a homomorphism.

(§ 11.1) Group Homomorphisms.

Def: A homomorphism between groups $(G, *)$ and (H, \circ) is a mapping $\phi: G \rightarrow H$, s.t. $\forall g_1, g_2 \in G$
 $\phi(g_1 * g_2) = \phi(g_1) \circ \phi(g_2)$.

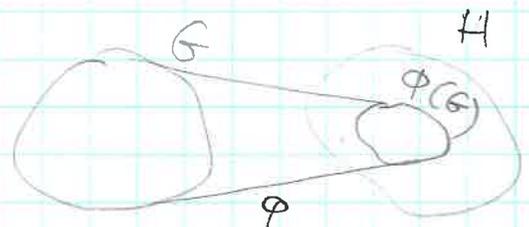
operation preservation!

Note: Range of ϕ in H is called the homomorphic image of ϕ .



Since ϕ is not bijective, then:

- Homomorphism is a weaker relationship between groups.



Examples: (1) Any isomorphism is a homomorphism that happens to be bijective.

(2) Consider $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ (under multiplication) and $GL_2(\mathbb{R})$, group of invertible 2×2 real matrices.

Then $\phi: GL_2(\mathbb{R}) \rightarrow \mathbb{R}^*$ defined by $\phi(A) = \det A$ $\forall A \in GL_2(\mathbb{R})$ is a homomorphism.

ϕ is well-defined and O.P.: $\phi(AB) = \det(AB) = (\det A)(\det B) = \phi(A)\phi(B)$
if $A=B \Rightarrow \phi(A) = \phi(B)$

(3) $\phi(x) = |x|; \mathbb{R}^* \rightarrow \mathbb{R}^*$ is a homomorphism:
 $\forall x, y \in \mathbb{R}^*, \phi(xy) = |xy| = |x||y| = \phi(x)\phi(y)$.

④ Group G , $g \in G$.
 Then $\varphi: \mathbb{Z} \rightarrow G$ by $\varphi(n) = g^n$ is a homomorphism:
 $\varphi(m+n) = g^{m+n} = g^m g^n = \varphi(m) \varphi(n)$

⑤ $\varphi: \mathbb{Z} \rightarrow \mathbb{Z}_n$ by $\varphi(m) = m \pmod{n}$ is a homomorphism.

⑥ $\varphi(x) = x^2; \mathbb{R} \rightarrow \mathbb{R}$ is not a homomorphism:
 $\varphi(a+b) = (a+b)^2 \neq a^2 + b^2 = \varphi(a) + \varphi(b)$
addition in \mathbb{R} !

But, $\varphi(x) = x^2; \mathbb{R}^* \rightarrow \mathbb{R}^*$ is a homomorphism:
 $\varphi(ab) = (ab)^2 = a^2 b^2 = \varphi(a) \varphi(b)$

⑦ Make sure φ is well-defined:

Consider \mathbb{Z}_6 and $\mathbb{Z}/3\mathbb{Z}$
 If we choose $\varphi: \mathbb{Z}/3\mathbb{Z} \rightarrow \mathbb{Z}_6$ defined by $\varphi(x + \langle 3 \rangle) = 3x \pmod{6}$, then it is not well-defined:
 O.P.: seems to be reasonable, as $3(x+y) = 3x + 3y$

$0 + \langle 3 \rangle = 3 + \langle 3 \rangle$, but $\varphi(0 + \langle 3 \rangle) = 3 \cdot 0 = 0$ in \mathbb{Z}_6
 coset of $3\mathbb{Z}$ in \mathbb{Z}
 while $\varphi(3 + \langle 3 \rangle) = 3 \cdot 3 = 3 \pmod{6}$ in \mathbb{Z}_6 .

⑧ Consider \mathbb{R} and $\mathbb{T} = \{z \in \mathbb{C}^* \mid |z| = 1\} \leq \mathbb{C}^* = \mathbb{C} \setminus \{0\}$
 the circle group
 Then $\varphi: \mathbb{R} \rightarrow \mathbb{T}$ defined by

$\varphi: \theta \mapsto e^{i\theta} = \cos\theta + i\sin\theta$

is a homomorphism. (See also Ex. 11.3 in Ch. 11)

Proposition (11.4) let G_1, G_2 be groups and (3)

let $\phi: G_1 \rightarrow G_2$ be a homomorphism. Then ϕ has the following properties:

- 1) $\phi(e_{G_1}) = e_{G_2}$
- 2) $\forall g \in G_1, \phi(g^{-1}) = [\phi(g)]^{-1}$
- 3) If $H_1 \leq G_1$, then $\phi(H_1) \leq G_2$
- 4) If $H_2 \leq G_2$, then $\phi^{-1}(H_2) = \{g \in G_1 \mid \phi(g) \in H_2\}$ is a subgroup of G_1 . If H_2 is normal in G_2 , then $\phi^{-1}(H_2)$ is normal in G_1 .

Proof: 1) see text / DIY

2) Let $g \in G_1$. Then $\phi(g^{-1})\phi(g) = \phi(g^{-1}g) = \phi(e_{G_1}) = e_{G_2}$
 $= \phi(gg^{-1}) = \phi(g)\phi(g^{-1}) \Rightarrow [\phi(g)]^{-1} = \phi(g^{-1})$.

3) Let $H_1 \leq G_1$. Then $e_{G_1} \in H_1 \Rightarrow \phi(e_{G_1}) = e_{G_2}$
 $\Rightarrow \phi(H_1) \neq \emptyset$. Take $x, y \in \phi(H_1)$, then $\phi(H_1)$

$\exists a, b \in H_1$, s.t. $\phi(a) = x$ & $\phi(b) = y$ and

$$xy^{-1} = \phi(a) [\phi(b)]^{-1} = \phi(a) \phi(b^{-1}) = \phi(\underbrace{ab^{-1}}_{\in H_1}) \in \phi(H_1).$$

Thus, by Prop. 3.31 (subgroup test),

$$\phi(H_1) \leq G_2.$$

4) Let $H_2 \leq G_2$. Consider $H_1 = \phi^{-1}(H_2) = \{g \in G_1, \phi(g) \in H_2\}$

Note $H_1 \neq \emptyset$ since for $g = e_{G_1}$, $\phi(e_{G_1}) = e_{G_2} \in H_2$.

Since $\forall g, h \in H_1$, $\phi(gh^{-1}) = \phi(g)\phi(h^{-1}) = \phi(g)[\phi(h)]^{-1} \in H_2$

By def. of H_1 , $gh^{-1} \in H_1 \Rightarrow H_1 \leq G_1$.

Now we show $H_1 = \Phi^{-1}(H_2) \triangleleft G_1$.

Consider ghg^{-1} for $\forall g \in G_1, \forall h \in H_1$.

Then $\Phi(ghg^{-1}) = \underbrace{\Phi(g)}_{\in G_2} \underbrace{\Phi(h)}_{\in H_2} \underbrace{[\Phi(g)]^{-1}}_{\in G_2}$. Since $H_2 \triangleleft G_2$

then $\forall x \in G_2 \cdot xH_2x^{-1} \subset H_2 \Rightarrow \Phi(ghg^{-1}) \in H_2$ Thm. 10.3

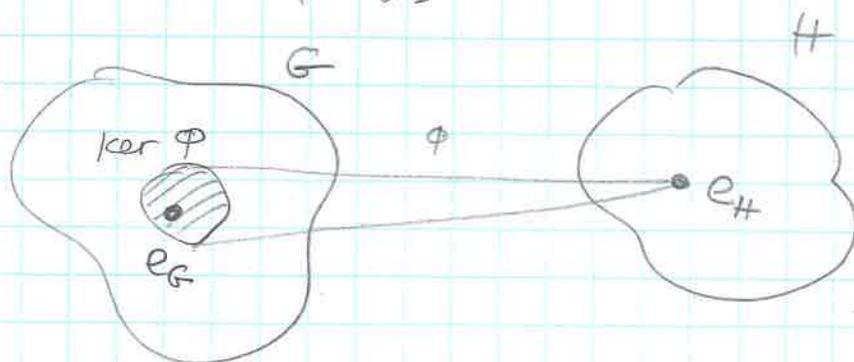
\Rightarrow by def. of H_1 , $ghg^{-1} \in H_1 \Rightarrow gH_1g^{-1} \subset H_1 \Rightarrow H_1 \triangleleft G_1$ Thm 10.3 \square

Def: Kernel of a homomorphism.

The kernel of a homomorphism $\Phi: \underbrace{G \rightarrow H}_{\text{groups}}$ is

the set $\ker \Phi = \{g \in G \mid \Phi(g) = e_H\}$.

$$(\ker \Phi = \Phi^{-1}(e_H))$$



Note:

$\ker \Phi \neq \emptyset$
since at
least

$$e_G \in \ker \Phi.$$

Theorem (11.5) Let $\Phi: \underbrace{G \rightarrow H}_{\text{groups}}$ be a homomorphism.

Then $\ker \Phi \triangleleft G$.

Proof: By Prop. 11.4, since $\{e_H\}$ is a trivial subgroup of H , then $\Phi^{-1}(e_H) = \{g \in G:$

$\Phi(g) = e_H\}$ is a subgroup of G . Since $\{e_H\} \triangleleft H$

then $\Phi^{-1}(e_H) = \ker \Phi \triangleleft G$.

Examples :

① $\phi : GL_2(\mathbb{R}) \rightarrow \mathbb{R}^* \text{ w/ } \phi(A) = \det A :$

$\ker \phi = \{A \in GL_2(\mathbb{R}) \mid \det A = 1\} = SL_2(\mathbb{R})$
special linear group

② $\phi : \mathbb{R}^* \rightarrow \mathbb{R}^* \text{ w/ } \phi(x) = |x| :$

$\ker \phi = \{\pm 1\}$

③ $\phi : \mathbb{R}^* \rightarrow \mathbb{R}^* \text{ w/ } \phi(x) = x^2 :$

$\ker \phi = \{\pm 1\}$

④ let $\mathbb{R}[x]$ be the group of all polynomials with real coefficients under addition.

(- Closure holds since if $f(x), g(x) \in \mathbb{R}[x] \Rightarrow f(x) + g(x) \in \mathbb{R}[x]$)

- Identity: $p(x) \equiv 0$ ($\forall f(x) \in \mathbb{R}[x]$, $f(x) + 0 = 0 + f(x) = f(x)$)

- Inverses: $\forall f(x) \in \mathbb{R}[x]$, $[f(x)]^{-1} = -f(x)$ since $f(x) + (-f(x)) = (-f(x)) + f(x) = 0$)

- Associativity: holds

Consider $\phi : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ defined by $\phi(f) = \frac{df}{dx}$.

It is a homomorphism, since ϕ is well-defined

(if $f=g \Rightarrow \frac{df}{dx} = \frac{dg}{dx}$) and operation preserving:

$\phi(f+g) = \frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx} = \phi(f) + \phi(g)$.

$\ker \phi = \{f \in \mathbb{R}[x] \mid \phi(f) = \frac{df}{dx} = 0\}$, that is

$\ker \phi$ is the set of constant polynomials. ($f=c$)

(5) $\varphi: \mathbb{R} \rightarrow \mathbb{C}^*$ defined by $\varphi(\theta) = e^{i\theta}$ is a homomorphism with

$$\ker \varphi = \{ \theta \in \mathbb{R} \mid \varphi(\theta) = e^{i\theta} = 1 \} = \{ 2\pi n, n \in \mathbb{Z} \}$$

$\cos \theta + i \sin \theta$

Note: $\ker \varphi \cong \mathbb{Z}$

(6) $\varphi: \mathbb{Z} \rightarrow G$ defined by $\varphi(n) = g^n$ for some $g \in G$. φ maps onto cyclic group generated by g .

φ is a homomorphism. Note:

- if $|g| = \infty \Rightarrow$ only $g^0 = e_G \Rightarrow \ker \varphi = \{0\}$

- if $|g| = n \Rightarrow g^n = e_G \Rightarrow \ker \varphi = \{ k \in \mathbb{Z}, g^k = e_G \}$
 $= n\mathbb{Z} = \{ \dots, -4n, -3n, -2n, -n, 0, n, 2n, 3n, \dots \}$

(7) Find all possible homomorphisms $\varphi: \mathbb{Z}_7 \rightarrow \mathbb{Z}_{12}$.

$$\mathbb{Z}_7 = \{0, 1, \dots, 6\} \xrightarrow{\varphi?} \mathbb{Z}_{12} = \{0, 1, \dots, 11\}$$

Note $\ker \varphi \triangleleft \mathbb{Z}_7$, and \mathbb{Z}_7 has only two trivial subgroups, $\{0\}$ and \mathbb{Z}_7 (since 7 is prime).

So, $\ker \varphi$ is $\{0\}$ or all of \mathbb{Z}_7 .

The image of \mathbb{Z}_7 , $\varphi(\mathbb{Z}_7)$ must be a subgroup of \mathbb{Z}_{12} . \mathbb{Z}_{12} can have subgroups of order

1, 2, 3, 4, 6, 12 only. So, φ cannot be one-to-one, because if φ was one-to-one, $\varphi(\mathbb{Z}_7)$ would have order 7.

Also: $\varphi(0) = \varphi(7) = \varphi(7 \cdot 1) = \varphi(1) + \dots + \varphi(1) = 7\varphi(1)$
(must) $= 0$ in \mathbb{Z}_{12} . Since $\overset{0 \pmod{7}}{\text{gcd}}(7, 12) = 1 \Rightarrow \varphi(1) = 0$ in \mathbb{Z}_{12}

$\Rightarrow \forall n \in \mathbb{Z}_7, \varphi(n) = n\varphi(1) = 0 \pmod{12} \Rightarrow \varphi$ must be the one that maps all $x \in \mathbb{Z}_7$ to zero in \mathbb{Z}_{12} .

⑧ Consider $\varphi: \mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12}$ defined by $\varphi(x) = 3x$

⑦

It is a homomorphism, since $\varphi(x+y) = 3(x+y)$
 $= 3x + 3y = \varphi(x) + \varphi(y)$. $\ker \varphi = \{ x \in \mathbb{Z}_{12}, 3x = 0 \}$
(mod 12) (mod 12)

$$= \{0, 4, 8\}.$$