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(§ 11.2)

The Isomorphism Theorems.

We can use factor groups to study

homomorphisms,

Def: The natural or canonicalhomomorphism is a map $\phi: G \rightarrow G/H$
defined by $\phi(g) = gH$. group \uparrow ($H \triangleleft G$)

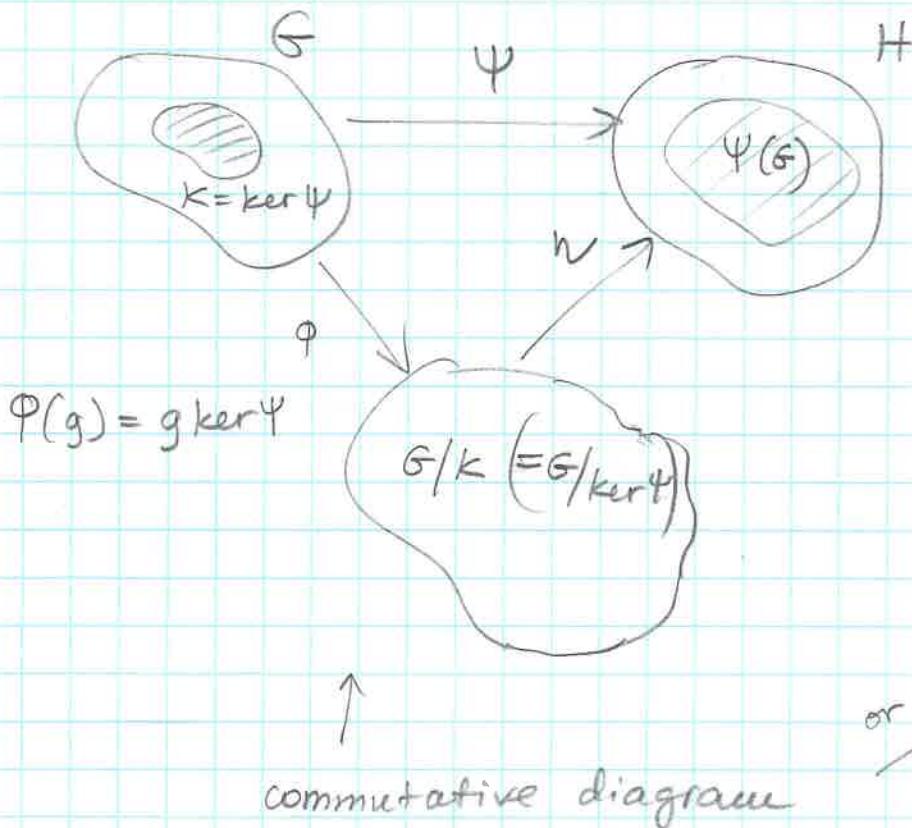
It is a homomorphism:

$$\phi(g_1g_2) = (g_1g_2)H = (g_1H)(g_2H) = \phi(g_1)\phi(g_2)$$

$$(\ker \phi = \{ g \in G \mid \phi(g) = H \} = H)$$

Theorem (11.10): First Isomorphism Theorem.

If $\psi: G \rightarrow H$ is a group homomorphism with $K = \ker \psi$, then $K \triangleleft G$. If $\phi: G \rightarrow G/K$ is the canonical homomorphism then there exists an isomorphism $\eta: G/K \rightarrow \psi(G)$ defined by $\eta = \psi \circ \phi$.



$$\begin{array}{ccc}
 G & \xrightarrow{\psi} & \psi(G) \subset H \\
 & \downarrow \phi & \downarrow \eta \\
 G/K & \xrightarrow{\quad} & \psi(G)
 \end{array}$$

or

$$\begin{array}{ccc}
 G & \xrightarrow{\psi} & \psi(G) \subset H \\
 & \downarrow \phi & \nearrow \eta \\
 G/\ker \psi & \xrightarrow{\quad} & \psi(G)
 \end{array}$$

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Example : cyclic

Consider $\Psi: \mathbb{Z} \rightarrow G = \langle g \rangle$ defined by $\Psi(n) = g^n$.
 Ψ is a homomorphism. cyclic

Note: Ψ is onto, since $\forall x \in G, \exists k \in \mathbb{Z}$ s.t.

$$\Psi(k) = g^k = x, \text{ i.e. } \Psi(k) = g.$$

(all multiples of m
are sent to e)

$$\text{If } |g| = m \Rightarrow g^m = e \Rightarrow \ker \Psi = m\mathbb{Z} \Rightarrow$$

$$\mathbb{Z}/\ker \Psi = \mathbb{Z}/m\mathbb{Z} \cong G. \text{ If } |g| = \infty, \text{ then } (g^0 = e)$$

$$\ker \Psi = \{0\} \text{ and } \mathbb{Z}/\{0\} = \mathbb{Z} \cong G$$

" $\Psi(z)$ "

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\Psi} & G \\ \varphi \searrow & \nearrow n & \\ \mathbb{Z}/\ker \Psi & = & \mathbb{Z}/\{0\} \\ \text{or } \mathbb{Z}/m\mathbb{Z} & & \end{array}$$

Two cyclic groups are isomorphic exactly when they have the same order.

$$|\mathbb{Z}/m\mathbb{Z}| = m = |G| \text{ or}$$

$$|\mathbb{Z}| = \infty = |G|$$

Proof of the theorem:

Thm. 11.5 $\Rightarrow K = \ker \Psi \triangleleft G$. Let $\Phi: G \rightarrow G/K$ be the canonical homomorphism and let

$\eta: G/K \rightarrow \Psi(G)$ be defined by $\eta(gK) = \Psi(g)$.

η is well-defined as if $g_1K = g_2K \Rightarrow$ by Lemma 63,

$g_2 \in g_1K \Rightarrow \exists k \in K = \ker \Psi$ s.t. $g_2 = g_1k \Rightarrow$

$$\begin{aligned} \eta(g_1K) &= \Psi(g_1) = \Psi(g_1) e_H = \Psi(g_1) \Psi(k) = \Psi(g_1k) \\ &\quad (\text{since } k \in \ker \Psi) \\ &= \Psi(g_2) = \eta(g_2K). \end{aligned}$$

Note γ is uniquely defined since $\Psi = \gamma \phi$.

(recall: $\phi(g) = gK : G \rightarrow G/K$, $\Psi(g) = \gamma(\phi(g)) = \gamma(\phi(g))$). Let us show γ is an isomorphism, that is, a homomorphism and a bijection:

$$\begin{aligned}\gamma(g_1 K g_2 K) &= \gamma(g_1 g_2 K) = \Psi(g_1 g_2) \stackrel{\text{OP}}{=} \Psi(g_1) \Psi(g_2) \\ &= \gamma(g_1 K) \gamma(g_2 K) \Rightarrow \gamma : G/K \rightarrow \Psi(G) \text{ is a homomorphism.}\end{aligned}$$

- γ is onto, since $\forall x \in \Psi(G) = \{\Psi(g), g \in G\}$ by definition, $\exists g \in G$ s.t. $\Psi(g) = x \Rightarrow$ there is a corresponding $gK \in G/K$ s.t. $\gamma(gK) = \Psi(g) = x$.
- γ is one-to-one, since if $\gamma(g_1 K) = \gamma(g_2 K)$
 $\Rightarrow \Psi(g_1) = \Psi(g_2) \Rightarrow [\Psi(g_1)]^{-1} \Psi(g_2) = e_H \Rightarrow$
 $[\Psi(g_1^{-1})] \Psi(g_2) = e_H \Rightarrow \Psi(g_1^{-1} g_2) = e_H \Rightarrow g_1^{-1} g_2 \in \underbrace{\ker \Psi}_{K}$
 $\Rightarrow g_1 K = g_2 K.$

Lemma 6.3

Thus, γ is an isomorphism between $G/K \cong \Psi(G)$. □

Another example: Consider groups $G \& H$ onto!

Then let $\Psi : G \times H \rightarrow H$ be defined by $\Psi(x, y) = y$
It's a homomorphism (check it!) w/ $\ker \Psi = \{(x, y) \in G \times H \mid y = e_H\}$
 $= \{(x, e) \mid x \in G\}$. By the 1st isomorphism theorem:

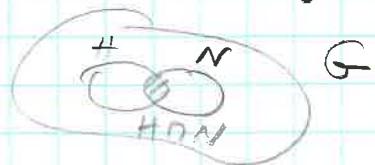
$$G \times H \xrightarrow{\Psi} H \quad \text{and} \quad G \times H / \ker \Psi \cong \Psi(G \times H) = H$$

(note: $\ker \Psi \cong G \Rightarrow G \times H / G \cong H$) □

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Theorem (11.12): Second Isomorphism Theorem

Let H be a subgroup of a group G and N be a normal subgroup of G . Then HN is a subgroup of G , and $HN \cap N$ is a normal subgroup of H , and $H/(HN \cap N) \cong HN/N$.



Note: $HN = \{hn \mid h \in H, n \in N\}$ ($\leq G$) ← show

$HN \cap N = \{g \in G \mid g \in H \text{ & } g \in N\}$ ($\leq H$) ← show

$\Psi: H \rightarrow HN/N$ by $h \mapsto hn$

↪ homomorphism (onto) ← show

1st Isomorphism Theorem :

$$H \xrightarrow{\Psi} HN/N = \varphi(H)$$

see
in text
proof

$$\begin{array}{ccc} \varphi & \searrow & \nearrow \psi \\ \text{canonical} & & \\ \text{homomorphism} & H/\ker \varphi = H/\{h \in H \mid h \in N\} = HN & \end{array}$$

$$\Rightarrow H/(HN \cap N) \cong HN/N$$

Theorem (11.13): Correspondence Theorem $\varphi(H) = H/N$

Let $N \triangleleft G$. Then $H \mapsto H/N$ is a one-to-one correspondence between the set of subgroups H containing N and the set of subgroups of G/N . Furthermore, the normal subgroups of G containing N correspond to normal subgroups of G/N .

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Theorem (11.13) \Rightarrow Theorem (11.14) Third

Isomorphism Theorem :

Let G be a group, $N \triangleleft G$, $H \triangleleft G$ and $N \subset H$.

Then $G/H \cong (G/N)/(H/N)$ ($\cong \frac{G/N}{H/N}$)

Example (text) : $\mathbb{Z}/m\mathbb{Z} \cong \underbrace{(\mathbb{Z}/mn\mathbb{Z})}_{\text{order } m} / \underbrace{(\mathbb{Z}/mn\mathbb{Z})}_{\text{order } mn} / \underbrace{(\mathbb{Z}/mn\mathbb{Z})}_{\text{order } n}$

$H = m\mathbb{Z} \triangleleft \mathbb{Z}$ \Rightarrow
 $N = mn\mathbb{Z} \triangleleft \mathbb{Z}$
& $N \subset H$