

(1)

(Chapter 13) The Structure of Groups.  
(briefly)

Theorem 13.4

Fundamental Theorem of  
Finite Abelian Groups.

Every finite abelian group  $G$  is isomorphic to a direct product of cyclic groups of the form:

$$G \cong \mathbb{Z}_{p_1^{d_1}} \times \mathbb{Z}_{p_2^{d_2}} \times \cdots \times \mathbb{Z}_{p_k^{d_k}}$$

where  $p_i$  are prime numbers, not necessarily distinct.

$d_i$ 's are  
uniquely  
determined  
by  $G$

Note: We call  $G$  a  $p$ -group if every element in  $G$  has as its order a power of  $p$ .

Examples:  $\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_8$  are 2-groups;  
 $\mathbb{Z}_3, \mathbb{Z}_9, \mathbb{Z}_{27}$  are 3-groups; and so on.

Thm. 13.4 is extremely powerful! We can construct an abelian finite group of any order!

Let  $n = p_1^{d_1} p_2^{d_2} \cdots p_k^{d_k}$ . Then form all groups of order  $p_1^{d_1}, p_2^{d_2}, \dots$ , and then form all possible external direct products of the groups.

Example:  $n = 1176 = 2^3 \cdot 3 \cdot 7^2 = (2 \cdot 2 \cdot 2) \cdot (3) \cdot (7 \cdot 7)$

The isomorphic classes of abelian groups of order 1176 are

$$\begin{aligned} & \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_7 \times \mathbb{Z}_7 \\ & \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_7 \times \mathbb{Z}_7 \\ & \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_7 \times \mathbb{Z}_7 \end{aligned}$$

(2)

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{49}$$

$$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{49}$$

$$\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_{49}$$

Order of $\mathbb{G}, p^k$	Partitions of $k \leq 4$	Possible products
prime $\rightarrow p \quad (k=1)$	1	$\{\mathbb{Z}_p\}$
$p^2 \quad (k=2)$	2, 1+1	$\{\mathbb{Z}_{p^2}, \mathbb{Z}_p \times \mathbb{Z}_p\}$
$p^3 \quad (k=3)$	3, 2+1, 1+1+1	$\{\mathbb{Z}_{p^3}, \mathbb{Z}_{p^2} \times \mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p\}$
$p^4 \quad (k=4)$	4, 2+2, 3+1, 2+1+1 1+1+1+1	$\{\mathbb{Z}_4, \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}, \mathbb{Z}_{p^3} \times \mathbb{Z}_p, \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p\}$
etc...		

↑ distinct isomorphic classes!

More examples:

1)  $n = 52 = 2^2 \cdot 13 = 2 \cdot 2 \cdot 13 : \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{13} \cong \mathbb{Z}_4 \times \mathbb{Z}_{13} \cong \mathbb{Z}_{52}$

2) Let  $\mathbb{G} = \{1, 8, 12, 14, 18, 21, 27, 31, 34, 38, 44, 47, 51, 53, 57, 64, 7\}$   
under multiplication mod 65.

$|\mathbb{G}| = 16 \Rightarrow \mathbb{G}$  is isomorphic to one of:

$\mathbb{Z}_4^4$

$$\begin{aligned} & \mathbb{Z}_{16} \\ & \mathbb{Z}_8 \times \mathbb{Z}_2 \\ & \mathbb{Z}_4 \times \mathbb{Z}_4 \\ & \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \\ & \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \end{aligned}$$

(in fact,  
 $\mathbb{G} \cong \mathbb{Z}_4 \times \mathbb{Z}_4$ )