

(§ 1.2) Sets & Equivalence Relations.

Set: a well-defined collection of objects
easy to determine whether an object in the set or not
called elements or members

Notation: $X = \{x_1, x_2, \dots, x_n\}$ or
 $X = \{x \mid x \text{ satisfies } P\}$
property

Example: $O = \{1, 3, 5, \dots\}$ odd #'s or $O = \{x \mid x > 0 \text{ is an even integer}\}$
 $9 \in O, 10 \notin O$

Important sets:
 $\mathbb{N} = \{1, 2, 3, \dots\}$ natural #'s
 $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ integers
 $\mathbb{Q} = \{p/q, p, q \in \mathbb{Z}, q \neq 0\}$ rational #'s
 $\mathbb{R} = \{x \mid x \text{ is a real number}\}$
 $\mathbb{C} = \{z \mid z \text{ is a complex number}\}$
 $z = a + bi, a, b \in \mathbb{R}$

Relations & Operations:

improper subset: $B \subset A$

• $A \subset B$ (or $B \supset A$): A is a subset of B

Ex: $\{1, 2, 3\} \subset \mathbb{Z}$; $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

(Note: every set is a subset of itself.)

• B is a proper set of A if $B \subset A, B \neq A$

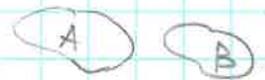
• $A \not\subset B$: A is not a subset of B.



• $A = B$ if $A \subset B$ and $B \subset A$

• Empty set \emptyset is a subset of every set.

• Union $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$



• Intersection $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$



Generally: $\bigcup_{i=1}^n A_i = A_1 \cup \dots \cup A_n$

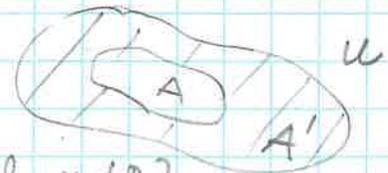
$\bigcap_{i=1}^n A_i = A_1 \cap \dots \cap A_n$

• Disjoint sets: $A \cap B = \emptyset$ (sets of even & odd #'s are disjoint)

• Sometimes we use the universal set U : a fixed set we work within.

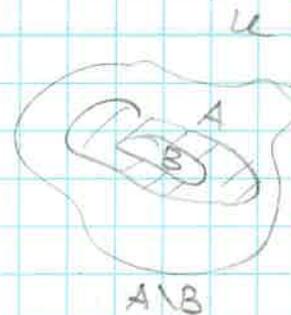
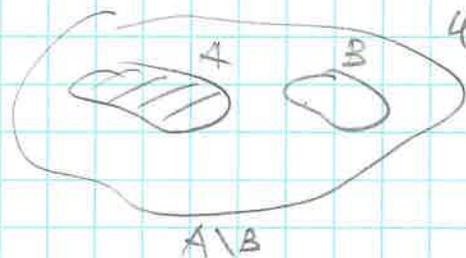
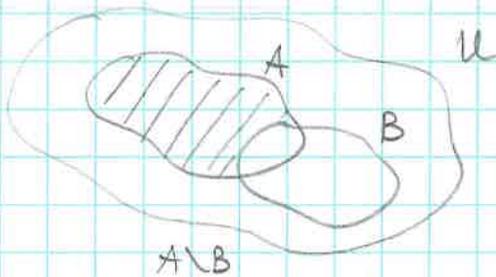
$\forall A \in U, A' = \{x \mid x \in U \text{ and } x \notin A\}$ is the complement of A .

(A' or A^c)



• Difference of A and B:

$A \setminus B = A \cap B' = \{x \mid x \in A \text{ and } x \notin B\}$



Proposition (1.2) Let A, B, C be sets. Then

① $A \cup A = A, A \cap A = A, A \setminus A = \emptyset \rightarrow$ idempotent laws

② $A \cup \emptyset = A, A \cap \emptyset = \emptyset \rightarrow$ identity laws

③ $A \cup (B \cap C) = (A \cup B) \cap C$ & $A \cap (B \cup C) = (A \cap B) \cup C$
 \rightarrow associativity laws

④ $A \cup B = B \cup A$ and $A \cap B = B \cap A \rightarrow$ commutativity laws

$$\begin{aligned} \textcircled{5} \quad A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \\ \textcircled{6} \quad A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \end{aligned} \quad \left. \vphantom{\begin{aligned} \textcircled{5} \\ \textcircled{6} \end{aligned}} \right\} \rightarrow \text{distributivity laws} \quad \textcircled{2}$$

Proof of ⑤ (Need to show $P=Q$, i.e., $P \subset Q$ and $Q \subset P$!)

$$\begin{aligned} \text{Start from } x \in A \cup (B \cap C) &\Rightarrow x \in A \text{ or } x \in B \cap C \\ \Rightarrow x \in A \text{ or } (x \in B \text{ and } x \in C) &\Rightarrow x \in A \cup B \text{ and } x \in A \cup C \\ \Rightarrow x \in (A \cup B) \cap (A \cup C). \end{aligned}$$

$$\text{So, } A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$$

$$\begin{aligned} \text{Now take } x \in (A \cup B) \cap (A \cup C) &\Rightarrow x \in A \cup B \text{ and } x \in A \cup C \\ \Rightarrow x \in A \text{ or } x \in B \text{ and } x \in A \text{ or } x \in C &\Rightarrow x \in A \text{ or } (x \in B \text{ and } x \in C) \\ \Rightarrow x \in A \cup (B \cap C), \text{ i.e. } (A \cup B) \cap (A \cup C) &\subseteq A \cup (B \cap C) \end{aligned}$$

So, the statement ⑤ is proved. \square

De Morgan's Laws (Thm. 1.3)

A, B sets.

$$\begin{aligned} \textcircled{1} \quad (A \cup B)' &= A' \cap B' \\ \textcircled{2} \quad (A \cap B)' &= A' \cup B' \end{aligned}$$

Proof of ①:

$$\text{Trivial case: } A \cup B = \emptyset \Rightarrow A = \emptyset, B = \emptyset \Rightarrow A' = B' = U, \quad \begin{array}{l} \text{universal} \\ \text{set} \\ \downarrow \end{array}$$

$$A' \cap B' = U = (A \cup B)'$$

In general: a) show $(A \cup B)' \subseteq A' \cap B'$.

$$\begin{aligned} x \in (A \cup B)' &\Rightarrow x \notin A \cup B \Rightarrow x \notin A \text{ and } x \notin B \Rightarrow \\ x \in A' \text{ and } x \in B' &\Rightarrow x \in A' \cap B'. \end{aligned}$$

b) show $A' \cap B' \subseteq (A \cup B)'$. Take $x \in A' \cap B' \Rightarrow$

④

$x \in A' \text{ and } x \in B' \Rightarrow x \notin A \text{ \& } x \notin B \Rightarrow x \notin A \cup B \Rightarrow$
 $x \in (A \cup B)'$. □

Cartesian Product of A & B:

$A \times B = \{ (a, b) : a \in A \text{ and } b \in B \}$ - set of ordered pairs

for A_1, \dots, A_n :

n-tuples

$A_1 \times \dots \times A_n = \{ (a_1, \dots, a_n) : a_i \in A_i, i=1, \dots, n \}$

$$A^n = \underbrace{A \times A \times \dots \times A}_n$$

Examples: 1) $A = \{1, 2, 3\}$, $B = \{a, b\} \Rightarrow$

$$A \times B = \{ (1, a), (1, b), (2, a), (2, b), (3, a), (3, b) \}$$

$$B \times A = \{ (a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3) \}$$

$A \times B \neq B \times A$ in general, unless $A = \emptyset, B = \emptyset$ or $A = B$.

2) \mathbb{R}^n is a set of n-tuples of real #'s
(n-dim. vectors)

Subsets of $A \times B$ are called relations.

A mapping (function) f from A to B is a relation ($f \subset A \times B$) where $\forall a \in A \exists ! b \in B$ s.t. $(a, b) \in f$. Notation: $f: A \rightarrow B$, $A \xrightarrow{f} B$

$$f(a) = b, \quad f: a \mapsto b.$$

A is the domain of f
 $f(A) = \{ f(a) \mid a \in A \} \subset B$ is the range / image of f .

Examples:

1) $f(x) = x^5 + 1$, Mapping ✓, $f: \mathbb{R} \rightarrow \mathbb{R}$, $(x, x^5 + 1) \in f$

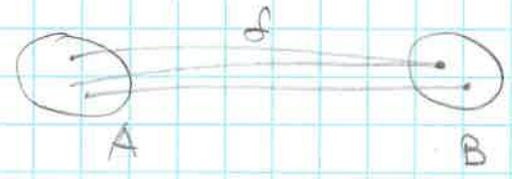
2) $f: \mathbb{Q} \rightarrow \mathbb{Z}$, s.t. $f(p/q) = p$. Mapping?

Nope: $f(1/2) \neq f(2/4)$ while $1/2 = 2/4$ (same) $\rightarrow 1?$
 (not well-defined!) | "2" $\rightarrow 2?$

3) Students $\xrightarrow{f?}$ Birthdays (mapping!)
 Birthdays $\xrightarrow{f?}$ Students (nope: $f(\text{Oct 1}) = \text{John}$
 $f(\text{Oct 1}) = \text{Mary}$)

• Onto or surjective mapping (map) $f: A \rightarrow B$;

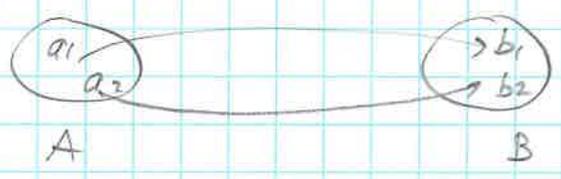
Range $f(A) = B$, i.e., each $b \in B$ is the image of at least one $a \in A$



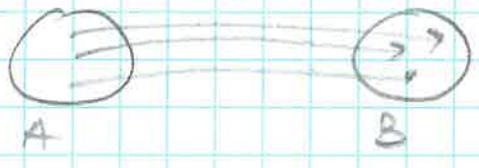
$(\forall b \in B \exists a \in A \text{ s.t. } f(a) = b)$

• One-to-one or injective mapping $f: A \rightarrow B$;

if $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$
 (or from $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$)



Onto & one-to-one together \Rightarrow bijective mapping (bijection)
 (Simply, perfect matching of A & B!)



Examples:

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1) $f(x) = 2x$ is a bijection

Onto: $\forall y \in \mathbb{R} \exists x = \frac{y}{2} \in \mathbb{R}$ s.t. $f(\frac{y}{2}) = 2 \cdot \frac{y}{2} = y$

One-to-one: $\forall y_1 = y_2 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$

2) $f: \mathbb{Z} \rightarrow \mathbb{Q}$ s.t. $f(n) = \frac{n}{1} = n$

One-to-one? Yes: $f(n_1) = f(n_2) \Rightarrow \frac{n_1}{1} = \frac{n_2}{1} \Rightarrow n_1 = n_2$

Onto? No! E.g., $\frac{1}{2} \in \mathbb{Q}$, $\frac{1}{2} \neq f(n) \forall n \in \mathbb{Z}$

(There are elements in \mathbb{Q} that are not images of \mathbb{Z})

3) $g: \mathbb{Q} \rightarrow \mathbb{Z}$ s.t. $g(\underbrace{p/q}_{\text{in } \mathbb{Q}}) = p$, $q > 0$, $p \geq q$ in lowest terms!

Onto? Yes: $\forall p \in \mathbb{Z}, \exists p/q \in \mathbb{Q}$ (for some q)

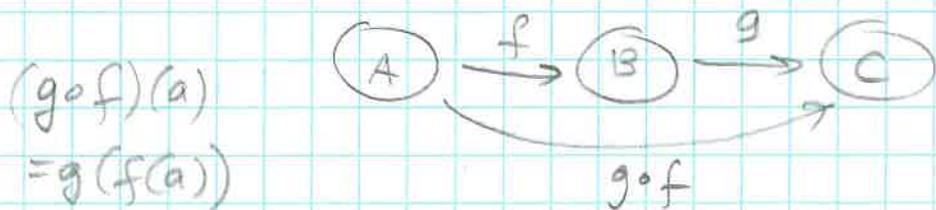
One-to-one? No: consider $g(p/q_1) = p = g(p/q_2) \Rightarrow p/q_1 = p/q_2$

4) $f(x) = \frac{1}{x}$, $f: \mathbb{R}_{\neq 0} \rightarrow \mathbb{R}_{\neq 0}$

bijection! (DIY) \rightarrow all nonzero real #'s

Composition mappings

If $f: A \rightarrow B$, $g: B \rightarrow C$, then $g \circ f: A \rightarrow C$



$g \circ f \neq f \circ g$ in general

Ex:

a) $f(x) = e^x$, $g(x) = \sin x \Rightarrow$

$$(f \circ g)(x) = f(g(x)) = e^{\sin x} \neq \sin(e^x) = (g \circ f)(x)$$

b) $f(x) = \sqrt[3]{x}$, $g(x) = x^3 \Rightarrow f \circ g = g \circ f$ identity function

More examples of mappings:

1) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2 \times 2}$ T_A send vector $\begin{pmatrix} x \\ y \end{pmatrix}$ to vector $A \begin{pmatrix} x \\ y \end{pmatrix}$

$T_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, s.t. $T_A(x, y) = (ax + by, cx + dy)$

$\forall (x, y) \in \mathbb{R}^2$, i.e., T_A -matrix multiplication:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

$T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear map (transformation)

2) Bijection from set S to itself is called a permutation of set S :

$S = \{1, 2, 3, 4, 5\}$, $\pi : S \rightarrow S$ s.t. $\pi(1) = 3, \pi(2) = 2, \pi(3) = 1, \pi(4) = 5, \pi(5) = 4$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \pi(1) & \pi(2) & \pi(3) & \pi(4) & \pi(5) \end{pmatrix}$$

Properties of Composition of maps (Thm. 1.5)

Let $f : A \rightarrow B, g : B \rightarrow C, h : C \rightarrow D$. Then

- (1) $(h \circ g) \circ f = h \circ (g \circ f)$ (associativity)
- (2) $g \circ f$ is one-to-one if f and g are both one-to-one
- (3) $g \circ f$ is onto if f and g are both onto.
- (4) $g \circ f$ is bijective if f and g are bijective.

Proof: Text has proofs of (1) & (3).

(4) follows from (2) & (3) Let us proof (2):

Let f, g be one-to-one mappings. Then if for some $x_1, x_2 \in A$, $(g \circ f)(x_2) = (g \circ f)(x_1)$, that is,

$$g(f(x_2)) = g(f(x_1)) \Rightarrow \begin{matrix} g \text{ is one-to-one} \\ f(x_2) = f(x_1) \end{matrix} \Rightarrow \begin{matrix} f \text{ is one-to-one} \\ x_2 = x_1 \end{matrix}$$



• Identity mapping: for any set S , id_S or id is a mapping $S \rightarrow S$ s.t. $\text{id}(s) = s \quad \forall s \in S$. (8)

• A mapping $g: B \rightarrow A$ is an inverse mapping of $f: A \rightarrow B$ if $g \circ f = \text{id}_A$ & $f \circ g = \text{id}_B$.

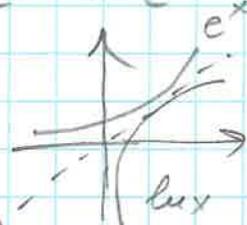
(g "undoes" f). g is usually written as f^{-1} .

• f is invertible if $\exists f^{-1}$.

Examples: (1) $f(x) = \ln x$ and $g(x) = e^x$ are inverses of each other

$$f(g(x)) = f(e^x) = \ln e^x = x$$

$$g(f(x)) = g(\ln x) = e^{\ln x} = x$$



(2) $A \in \mathbb{R}^{2 \times 2} \rightarrow$ set of real-valued 2×2 matrices. Consider $T_A(x, y) = (ax + by, cx + dy): \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Then $T_A^{-1} = T_{A^{-1}}$.

E.g., $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$, $T_A(x, y) = (3x + y, 5x + 2y)$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} \Rightarrow$$

$$T_A^{-1}(x, y) = T_{A^{-1}}(x, y) = (2x - y, -5x + 3y)$$

Check: $(T_A^{-1} \circ T_A)(x, y) = T_A^{-1}(T_A(x, y)) =$

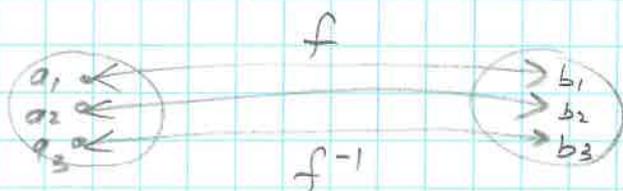
$$= T_A^{-1}(3x + y, 5x + 2y) = (2(3x + y) - (5x + 2y), -5(3x + y) + 3(5x + 2y))$$

$$= (6x + 2y - 5x - 2y, -15x - 5y + 15x + 6y) = (x, y) \quad \checkmark$$

[Check $(T_A \circ T_A^{-1})(x, y) = (x, y)$ yourself!]

Note: not every map has an inverse.

Thm. 1.20 A mapping is invertible if and only if it is bijective (one-to-one and onto)



Proof: if and only if, i.e. \Leftrightarrow (9)

\Rightarrow $f: A \rightarrow B$ invertible, $g = f^{-1}: B \rightarrow A$. Let us show f is bijective.

Note: $g \circ f = id_A$, i.e. $g(f(a)) = a \forall a \in A$.

If for $a_1, a_2 \in A$, $f(a_1) = f(a_2)$, then $a_1 = g(f(a_1))$

$= g(f(a_2)) = a_2 \Rightarrow a_1 = a_2 \Rightarrow f$ is one-to-one.

Onto? Suppose $b \in B$, we need to show $\exists a \in A$ s.t. $f(a) = b$. If $b \in B \Rightarrow g(b) \in A$ w/ $f(g(b)) = b$. By letting $a = g(b)$ for any $b \in B$, we have $f(a) = b \Rightarrow f$ is onto.

\Leftarrow Let f be bijective and let $b \in B$. Since f is onto, $\exists a \in A$ s.t. $f(a) = b$. Since f is one-to-one, a is unique ($\nexists a' \neq a$ s.t. $f(a') = f(a)$).

\Rightarrow if we define g by $g(b) = a \Rightarrow$ the inverse of f is constructed, $g = f^{-1}: B \rightarrow A$. \square

Equivalence Relations & Partitions

Def: An equivalence relation on a set X is a relation $R \subset X \times X$, s.t.

- ① $(x, x) \in R \forall x \in X$ (reflexive property)
- ② $(x, y) \in R \Rightarrow (y, x) \in R$ (symmetric property)
- ③ (x, y) and $(y, z) \in R \Rightarrow (x, z) \in R$ (transitive property)

Notation: instead of $(x, y) \in R$: $x \sim y$

Examples:

- 1) Clearly, for $p, q, r, s \in \mathbb{Z}$, $q \neq 0, s \neq 0$,
 $\frac{p}{q} \sim \frac{r}{s}$ if $ps = qr$ (① $\frac{p}{q} \sim \frac{p}{q}$, ② $\frac{p}{q} \sim \frac{r}{s} \Rightarrow \frac{r}{s} \sim \frac{p}{q}$,
③ $\frac{p}{q} \sim \frac{r}{s}, \frac{r}{s} \sim \frac{t}{u} \Rightarrow ps = rq, ru = ts \Rightarrow psu = rqu = \underbrace{rqu}_{ts} = tsq \Rightarrow$
 $pus = tq \Rightarrow \frac{p}{q} \sim \frac{t}{u}$)

2) in \mathbb{Z} , $m \sim n \Leftrightarrow |m| = |n|$

Indeed,

① $n \sim n$ as $|n| = |n|$

② if $|m| = |n| \Rightarrow m \sim n$ or $n \sim m$

③ if $|m| = |n| = |p| \Rightarrow m \sim p$
 $m \sim n$ $n \sim p$

3) $f \sim g$ (differentiable functions on \mathbb{R}) if $f'(x) = g'(x)$ (see text)

4) $A, B \in \mathbb{R}^{2 \times 2}$. We define $A \sim B$ if \exists invertible $P \in \mathbb{R}^{2 \times 2}$ s.t. $PAP^{-1} = B$ (similarity)

① $A \sim A$ since $IAI^{-1} = IA = A$ where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

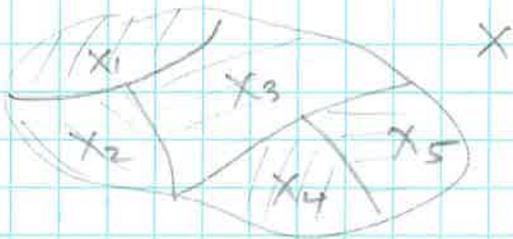
② Let $A \sim B$, i.e. $\exists P$ s.t. $PAP^{-1} = B$, P is invertible
 $\Rightarrow \underbrace{P^{-1}(PAP^{-1})P}_{I} = \underbrace{P^{-1}BP}_{I} \Rightarrow P^{-1}BP = A$ and $B \sim A$
 $(P^{-1})B(P^{-1})^{-1} = A$ w/ P^{-1} .

③ Let $A \sim B$ and $B \sim C$ s.t. $PAP^{-1} = B$ and $QBQ^{-1} = C$ for some P & Q . Then

$$C = QBQ^{-1} = QPAP^{-1}Q^{-1} = (QP)A(QP)^{-1} \Rightarrow A \sim C \text{ w/ matrix } QP.$$

Partition P of a set X is a family of nonempty subsets X_1, X_2, \dots of X such that

$$X_i \cap X_j = \emptyset \text{ for } i \neq j \text{ (disjoint) \& } \bigcup_i X_i = X.$$



If \sim is an equivalence relation on X , then

$[x] = \{y \in X : y \sim x\}$ is called the equivalence class of x .

Equivalence relation gives rise to a partition via equivalence classes.

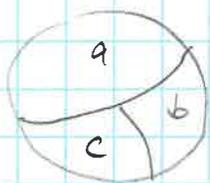
Examples: 1) (p, q) & (r, s) are in the same equivalence class. In \mathbb{Z} if $\frac{p}{q} \sim \frac{r}{s}$ ($ps = qr$), i.e. if $\frac{p}{q} = \frac{r}{s} = \frac{a}{b}$ in lowest terms.

2) f & g (diff.) are in the same equivalence class if $f'(x) = g'(x)$ ($f \sim g$)
 ($f = g + C$)

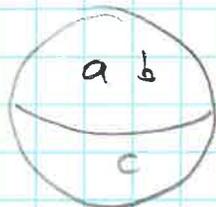
Thm. 1.25: Given an equivalence relation \sim on a set X , the equivalence classes form a partition of X . Conversely, if $P = \{X_i\}$ is a partition of X , then there exists an equivalence relation on X w/ equivalence classes X_i .
 (See proof in text) ($\sim \Rightarrow \sim$ classes \Rightarrow partition or $P = \{X_i\} \Rightarrow X_i$'s are equiv. classes)

Corollary: Two equivalence classes of an equivalence relation are either disjoint or equal.

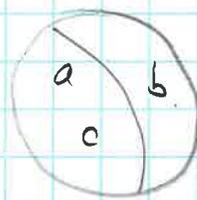
Examples: 1) $A = \{a, b, c\} \Rightarrow$ 5 ways of partitioning:



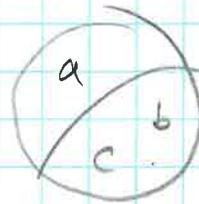
$a \sim a$
 $b \sim b$
 $c \sim c$



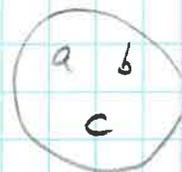
$a \sim a$
 $b \sim b$
 $c \sim c$
 $a \sim b$



$a \sim a$
 $b \sim b$
 $c \sim c$
 $a \sim c$



$a \sim a$
 $b \sim b$
 $c \sim c$
 $b \sim c$



$a \sim a$
 $b \sim b$
 $c \sim c$
 $a \sim b \sim c$

2) Consider integers r and s and let $n \in \mathbb{N}$

We say r is congruent to s modulo n ($\text{mod } n$)

if $r - s = nk$ for some $k \in \mathbb{Z}$: $r \equiv s \pmod{n}$

Ex: $43 \equiv 16 \pmod{9}$ as $43 - 16 = 27$, divisible by 9.

Also: $40 \equiv 0 \pmod{2}$

It is an equivalence relation of \mathbb{Z} : $r \equiv s \pmod{n}$

① $r \sim r$ since $r - r = 0$, divisible by n

② if $r \sim s \Rightarrow r - s = nk = -(s - r) \Rightarrow s - r = n(-k)$,
 i.e. divisible by $n \Rightarrow s \equiv r \pmod{n} \Rightarrow s \sim r$.

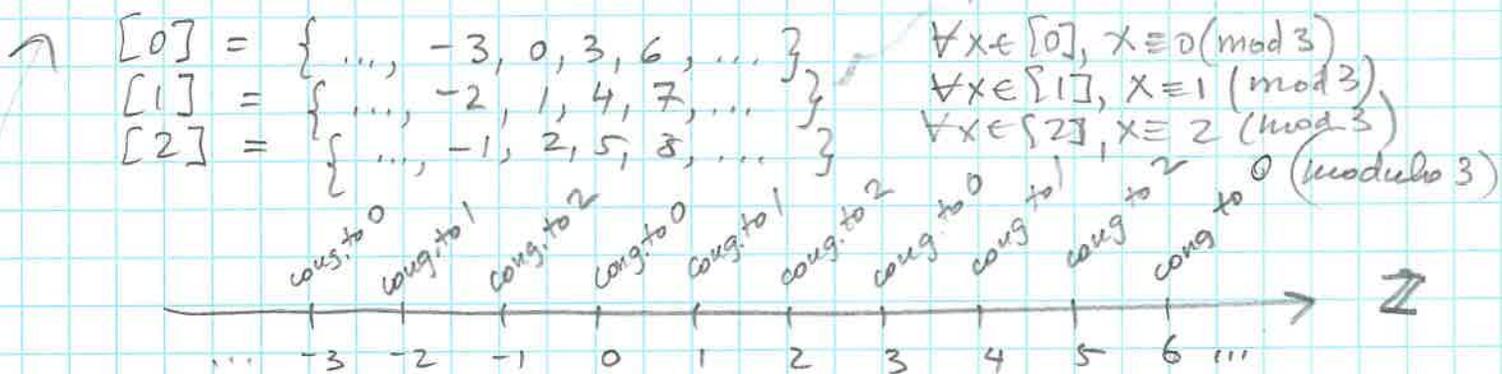
$$\textcircled{3} \quad r \sim s \text{ and } s \sim t \Rightarrow \begin{matrix} r-s = nk & k, l \in \mathbb{Z} \\ s-t = nl \end{matrix} \quad \textcircled{12}$$

$$\Rightarrow r-t = r-s+s-t = nk+nl = n(k+l) \Rightarrow r \equiv t \pmod{n} \Rightarrow r \sim t.$$

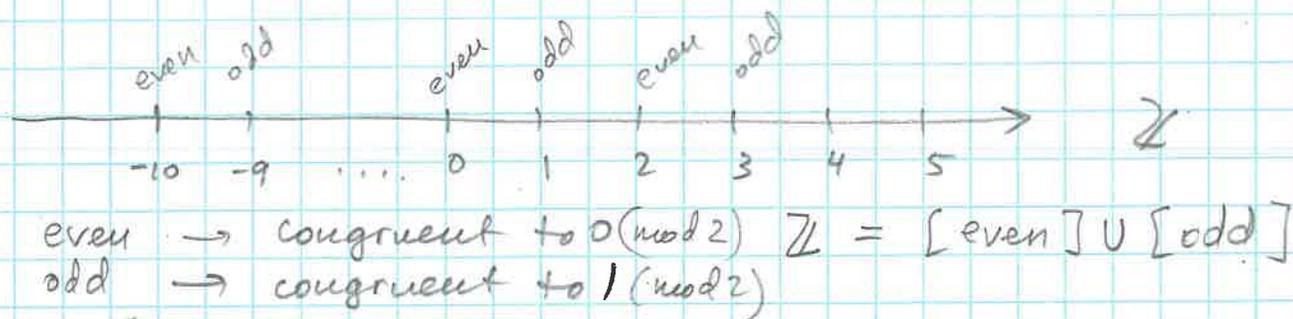
Consider $r \sim s$ on \mathbb{Z} via $r \equiv s \pmod{3}$, i.e.

$$r-s = 3k, k \in \mathbb{Z} \Rightarrow \mathbb{Z} = [0] \cup [1] \cup [2] \text{ with partition}$$

(or $3 \mid r-s$)



Similarly: if we choose $r \equiv s \pmod{2}$ as an equivalence relation on \mathbb{Z} :



Also: $r \equiv s \pmod{n}$ means $n \mid r-s$
 "n divides r-s"