§ 10.5 Romberg Integration

Richardson extrapolation:

$$f(x) = \varphi_0(h) + \underbrace{O(h)}_{a_1 h + a_2 h^2 + \dots}$$

then $f(x) = \varphi_0(\frac{h}{2}) + O(\frac{h}{2})$ $\Rightarrow$

find $f(x) = \underbrace{\varphi_1(h)}_{\text{new approx}} + O(h^2)$, φ_1 is in terms of $\varphi_0(h)$ & $\varphi_0(\frac{h}{2})$

then repeat...

We can apply this idea to integration:

Start from the composite trapezoid rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f_0 + 2f_1 + 2f_2 + \dots + 2f_{n-1} + f_n\right] \underbrace{+ Ch^2 + O(h^4)}_{O(h^2)}$$

$f_i = f(x_i)$

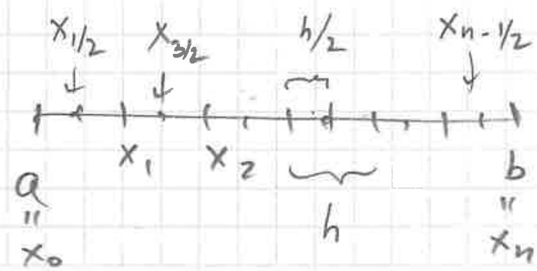
(p. 233, § 10.2)

(no odd powers of h ...)

Define $T_h = \frac{h}{2}\left[f_0 + 2f_1 + \dots + 2f_{n-1} + f_n\right]$

$$\Rightarrow \int_a^b f(x)\,dx = T_h + Ch^2 + O(h^4)$$

Next, $\int_a^b f(x)\,dx = \frac{h}{4}\left[\overbrace{f_0 + 2f_{1/2} + 2f_1 + \dots + 2f_{n-1/2}}^{\text{more terms}} + f_n\right] + \frac{Ch^2}{4} + O(h^4)$



$$T_{h/2} = \frac{h}{4}\left[f_0 + 2f_{1/2} + 2f_1 + \dots + 2f_{n-1/2} + f_n\right]$$

To eliminate $O(h^2)$ term, form

$$\int_a^b f(x)\,dx = \frac{4}{3}T_{h/2} - \frac{1}{3}T_h + O(h^4) =$$

$$= \frac{h}{3}\left[f_0 + 2f_{1/2} + 2f_1 + \dots + 2f_{n-1/2} + f_n\right]$$

$$- \frac{h}{6}\left[f_0 + 2f_1 + \dots + 2f_{n-1} + f_n\right] + O(h^4)$$

$$= \underbrace{\frac{h}{6}\left[f_0 + 4f_{1/2} + 2f_1 + \dots + 2f_{n-1} + 4f_{n-1/2} + f_n\right]}_{\text{but this is ... the composite Simpson's rule!}} + O(h^4)$$

This process can be repeated! It is called Romberg integration.

<u>Note:</u> MATLAB routine <u>romberg.m</u> is available, see p. 243.

---

§10.6 on periodic functions is optional to read.