(1)

§10.7 Singularities (Improper Integrals).

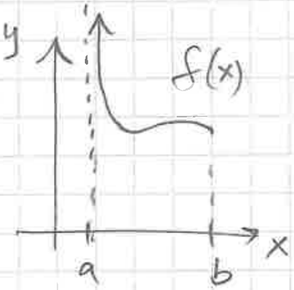
$\int_a^b f(x)\,dx$ ⇐ apply num. integration tools if f is smooth.

What if 1) $f \to \pm\infty$ at a or b?

2) f has discontinuity inside $[a,b]$?

3) ~~or~~ we need to evaluate $\int_a^\infty f(x)\,dx$?

- **Left Endpoint Singularity:**



$\lim_{x\to a^+} f(x) = \pm\infty \Rightarrow \int_a^b f(x)\,dx$ is improper

If $f(x)$ can be written as

$$f(x) = \frac{g(x)}{(x-a)^p}, \quad 0<p<1 \text{ and}$$

$g(x)$ has sufficiently many continuous derivatives on $[a,b]$, then $\int_a^b f(x)\,dx = \int_a^b \frac{g(x)}{(x-a)^p}\,dx$

First, consider $g(x)\equiv 1$. Then

$$\int_a^b \frac{dx}{(x-a)^p} \text{ converges } \Leftrightarrow 0<p<1 :$$

$$\int_a^b \frac{dx}{(x-a)^p} = \lim_{t\to a^+}\int_t^b \frac{dx}{(x-a)^p} = \lim_{t\to a^+}\left[\frac{(x-a)^{1-p}}{1-p}\right]_{x=t}^{x=b}$$

$$= \lim_{t\to a^+}\left[\frac{(b-a)^{1-p}}{1-p} - \underbrace{\frac{(t-a)^{1-p}}{1-p}}_{\downarrow\, 0}\right] = \frac{(b-a)^{1-p}}{1-p}$$

Now, back to $f(x) = \dfrac{g(x)}{(x-a)^p}$ and

$$\int_a^b f(x)\,dx = \int_a^b \frac{g(x)}{(x-a)^p}\,dx \underset{\substack{\delta>0 \\ \text{small}}}{=} \underbrace{\int_a^{a+\delta} \frac{g(x)}{(x-a)^p}\,dx}_{?}$$

$$+ \underbrace{\int_{a+\delta}^{b} \frac{g(x)}{(x-a)^p}\,dx}_{\text{apply any num. Integration techique!}}$$

$$\int_a^{a+\delta} \frac{g(x)}{(x-a)^p}\,dx \underset{\substack{\text{Taylor's} \\ \text{formula}}}{=} \int_a^{a+\delta} \left[\frac{g(a)}{(x-a)^p} + g'(a)(x-a)^{1-p} + \frac{g''(a)}{2}(x-a)^{2-p} \right.$$

$$\left. + \frac{g'''(a)}{6}(x-a)^{3-p} + \dots \right] dx$$

$$= g(a)\frac{\delta^{1-p}}{1-p} + g'(a)\frac{\delta^{2-p}}{2-p} + g''(a)\frac{\delta^{3-p}}{3-p} + \dots$$

<u>Example:</u> $\displaystyle\int_0^1 \frac{e^x}{\sqrt{x}}\,dx$

$a=0,\ b=1,\ p=1/2$

$\displaystyle\lim_{x\to 0^+} \frac{e^x}{\sqrt{x}} = \infty$

$g(x)=e^x$

Also: $\underset{\substack{\text{about} \\ 0}}{e^x} = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dots$

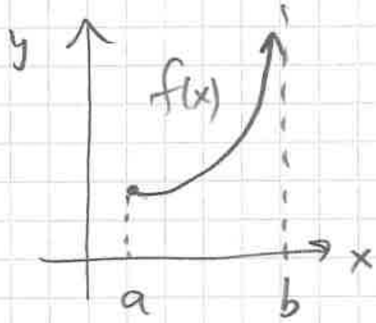
$$\int_0^1 \frac{e^x}{\sqrt{x}}\,dx = \underbrace{\int_0^{\delta} \frac{e^x}{\sqrt{x}}\,dx}_{} + \underbrace{\int_{\delta}^{1} \frac{e^x}{\sqrt{x}}\,dx}_{\text{use any tool}}$$

$$\int_0^{\delta} \frac{e^x}{\sqrt{x}}\,dx = g(0)\frac{\delta^{1-1/2}}{1-1/2} + g'(0)\frac{\delta^{2-1/2}}{2-1/2} + g''(0)\frac{\delta^{3-1/2}}{3-1/2} + \dots$$

$$= 2\sqrt{\delta} + \frac{2}{3}\delta^{3/2} + \frac{2}{5}\delta^{5/2} + \dots$$

(Choose # terms)

Right Endpt Singularity



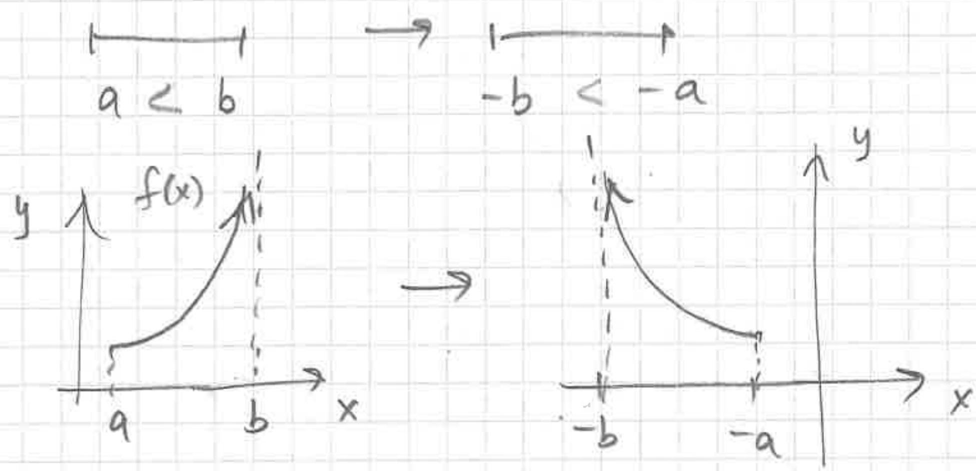
If $\lim_{x \to b^-} f(x) = \pm\infty$, then

do substitution: $y = -x$

(change of variable) $dy = -dx$

$$\Rightarrow \int_a^b f(x)\,dx = \int_{-a}^{-b} f(-y)(-dy) = \int_{-b}^{-a} f(-y)\,dy$$

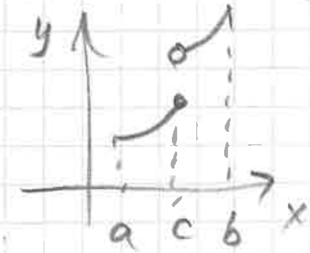
has singularity at the left endpt. $-b$



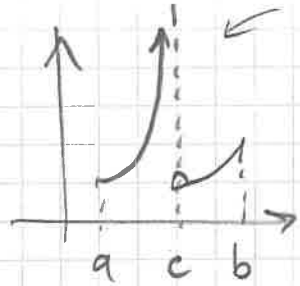
Use the idea described above.

Discontinuity inside $[a,b]$:

If $f(x)$ has discontinuity at $c \in (a,b)$ then split:



$$\int_a^b f(x)\,dx = \underbrace{\int_a^c f(x)\,dx + \int_c^b f(x)\,dx}_{\text{use tools from numerical integration}}$$

If  ← $f \to \pm\infty$ at c

then split and apply technique described above for left (or right) endpt singularity.

- <u>Infinite Singularity</u>: let f be smooth.

$$\underbrace{\int_a^{\infty} f(x)\,dx}_{\text{numerically?}} = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

$$\int_a^{\infty} f(x)\,dx = \underbrace{\int_a^b f(x)\,dx}_{\text{for } b \text{ large}} + \underbrace{\int_b^{\infty} f(x)\,dx}_{\text{negligible!}} \simeq \int_a^b f(x)\,dx$$

<u>or</u>

we will make substitution: $\xi = \frac{1}{x}$, $x = \frac{1}{\xi}$, $dx = -\frac{1}{\xi^2}\,d\xi \Rightarrow$

$$\int_b^{\infty} f(x)\,dx = \int_{1/b}^{0} f\left(\tfrac{1}{\xi}\right)\left(-\tfrac{1}{\xi^2}\right)d\xi = \int_0^{1/b} f\left(\tfrac{1}{\xi}\right)\xi^{-2}\,d\xi$$

If $\lim_{\xi\to 0^+} f\left(\frac{1}{\xi}\right)\xi^{-2}$ is finite, then we can use any standard integration tool; if not, then the integral can be handled as one w/ left endpt singularity, as above.

$$\left(\frac{f(\xi)}{\xi^2} \underset{0<p<1}{=} \frac{g(\xi)}{\xi^p} \text{ for some } g(\xi), \ldots \right)$$

Example: $I = \int_1^\infty x^{-3/2} \sin\left(\frac{1}{x}\right) dx \approx ?$

Let $\xi = \frac{1}{x}$, then $d\xi = -\frac{1}{x^2} dx$

and $dx = -x^2 d\xi = -\frac{1}{\xi^2} d\xi \Rightarrow$

$$\int_1^\infty x^{-3/2} \sin\left(\frac{1}{x}\right) dx = \int_1^0 \left(\frac{1}{\xi}\right)^{-3/2} \left(-\frac{1}{\xi^2}\right) \sin\xi \, d\xi$$

$$= \int_0^1 \underbrace{\frac{\sin\xi}{\sqrt{\xi}}}_{\substack{\downarrow \text{ as } \xi \to 0^+ \\ 0}} d\xi$$

Note: $\sin\xi = \underbrace{\xi - \frac{\xi^3}{3!}}_{\substack{\text{4th Taylor's polynomial } P_4(\xi) \\ \text{about } \xi_0 = 0}} + \dots$

So, $I \approx \int_0^1 \frac{\xi - \frac{\xi^3}{6}}{\sqrt{\xi}} d\xi = \int_0^1 \left(\xi^{1/2} - \frac{1}{6}\xi^{5/2}\right) d\xi$

$$= \left[\frac{2}{3}\xi^{3/2} - \frac{1}{21}\xi^{7/2}\right]_0^1 = \frac{2}{3} - \frac{1}{21} \approx 0.619$$