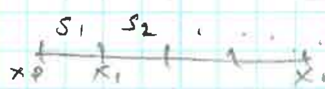
§ 8.6.2. Cubic Spline Interpolation.

Recall: A piecewise cubic Hermite interpolant has one continuous derivative. To obtain smoother interpolants, consider another option:

- A cubic spline interpolant $S(x)$ is a piecewise cubic that interpolates a func. f at the nodes $x_0, \dots, x_n$ (also called knots) and has two continuous derivatives, $S'(x)$ and $S''(x)$.

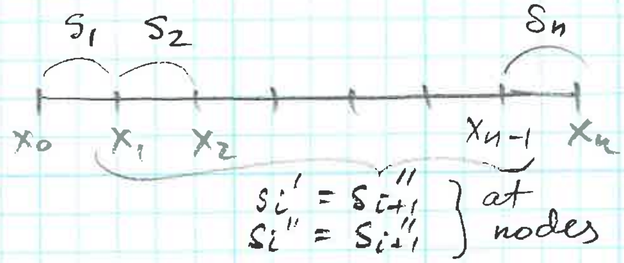
To get the formula for S, we will solve a global linear system (instead of writing S, S', S'' on each $[x_{i-1}, x_i]$ separately).

- What is the number of equations and unknowns in this system?
  - A cubic S has 4 parameters (coefficients) for each piece on each subinterval $[x_{i-1}, x_i]$ $\Rightarrow$ $4n$ unknowns (n = # subintervals)



  - From $S(x_i) = f(x_i)$, $i = 0, 1, \dots, n$ we get $2n$ equations, since in each subinterval we have to match f at 2 endpoints.
  - Also: S' and S'' must be continuous at the knots. We choose $x_1, \dots, x_{n-1}$ knots, leaving x_0, x_n without this requirement, this will allow us to enforce additional conditions on these knots later (results in

different cubic splines). Thus, by requiring s', s'' to be continuous at $x_1, \ldots, x_{n-1}$, we get $2(n-1) = 2n-2$ equations:



if s_i is a cubic on the ith subinterval, then:

$$s_i'(x_i) = s_{i+1}'(x_i)$$
$$s_i''(x_i) = s''_{i+1}(x_i)$$
$$i = 1, 2, \ldots, n-1$$

Total. $4n$ unknowns

$$(2n-2) + 2n = 4n-2 \text{ equations}$$

$$\Downarrow$$

2 degrees of freedom.

Derivation of s:

Let $z_i = s''(x_i)$, $i = 1, \ldots, n-1$. If we suppose that z_i's are "known", then linear s'' must satisfy

$$s''_i(x) = \underbrace{z_{i-1}\frac{x - x_i}{x_{i-1} - x_i} + z_i \frac{x - x_{i-1}}{x_i - x_{i-1}}}_{\text{Lagrange form}} \quad \text{in } [x_{i-1}, x_i].$$

w/ $h = x_i - x_{i-1}$ for all i

ith piece of our piecewise s''

$$\left(s_i''(x_i) = s''_{i+1}(x_i) = z_i,\ i = 1, \ldots, n-1 \right)$$

So, $s''_i(x) = \dfrac{z_{i-1}}{h}(x_i - x) + \dfrac{z_i}{h}(x - x_{i-1})$. Now integrate twice to get:

$$s_i'(x) = -\frac{z_{i-1}}{h}\frac{(x_i - x)^2}{2} + \frac{z_i}{h}\frac{(x - x_{i-1})^2}{2} + C_i, \text{ and}$$

$$s_i(x) = \frac{z_{i-1}}{h}\frac{(x_i - x)^3}{6} + \frac{z_i}{h}\frac{(x - x_{i-1})^3}{6} + C_i(x - x_{i-1}) + D_i$$

$$\left(s_i'(x) = \int_{x_{i-1}}^{x} s_i''(t)\,dt + C_i \ \& \ s_i(x) = \int_{x_{i-1}}^{x} s_i'(t)\,dt + D_i \right)$$

by the FTC

Recall: $s_i(x_{i-1}) = f(x_{i-1}) = f_{i-1}$

(for simpicity of notation)

$$s_i(x_{i-1}) = z_{i-1}\cdot\frac{h^2}{6} + D_i = f_{i-1} \Rightarrow D_i = f_{i-1} - \frac{z_{i-1}h^2}{6};$$

now from $s_i(x_i) = f(x_i) = f_i$, we have:

$$s_i(x_i) = z_i\frac{h^2}{6} + C_i h + D_i = \frac{z_i h^2}{6} + C_i h + \overbrace{f_{i-1} - \frac{z_{i-1}h^2}{6}}^{D_i} = f_i$$

$$\Rightarrow C_i = \frac{1}{h}\left[f_i - f_{i-1} + \frac{h^2}{6}(z_{i-1} - z_i)\right].$$

So, $s_i(x)$ (cubic piece on the ith interval)

$$s_i(x) = \frac{1}{h}z_{i-1}\frac{(x_i - x)^3}{6} + \frac{1}{h}z_i\frac{(x - x_{i-1})^3}{6} + \frac{1}{h}\left[f_i - f_{i-1} + \frac{h^2}{6}(z_{i-1} - z_i)\right](x - x_{i-1}) + f_{i-1} - \frac{h^2}{6}z_{i-1}$$

Once we know $z_1, \dots, z_{n-1} \Rightarrow S$ is known anywhere.

Using continuity of s' (which we did not use yet):

$$s_i'(x_i) = s_{i+1}'(x_i) \qquad (i = 1, \dots, n-1)$$

$$z_i\cdot\frac{h}{2} + \frac{1}{h}(f_i - f_{i-1}) + \frac{h}{6}(z_{i-1} - z_i) = -\frac{h}{2}z_i + \frac{1}{h}(f_{i+1} - f_i) + \frac{h}{6}(z_i - z_{i+1})$$

$i = 1, \dots, n-1.$

Taking z_i's to the L.H.S. and f_i's to the R.H.S:

$$\frac{2h}{3}z_i + \frac{h}{6}z_{i-1} + \frac{h}{6}z_{i+1} = -\frac{2}{h}f_i + \frac{1}{h}f_{i-1} + \frac{1}{h}f_{i+1}, \quad i = 1, \dots, n-1.$$

We can rewrite this as a symmetric tridiagonal linear system for $z_1, \dots, z_{n-1}$:

$$\downarrow \qquad \downarrow$$

$$S''(x_1), \dots, S''(x_{n-1})$$

$$\begin{pmatrix} \alpha_1 & \beta_1 & & & & \\ \beta_1 & \alpha_2 & \beta_2 & & 0 & \\ & \beta_2 & \alpha_3 & \beta_3 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & & & \beta_{n-2} \\ & & & & \beta_{n-2} & \alpha_{n-1} \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ \vdots \\ z_{n-1} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ \vdots \\ b_{n-1} \end{pmatrix} \Rightarrow \underline{\underline{z_i\text{'s}}}$$

where $\alpha_i = \frac{2h}{3}, \ \beta_i = \frac{h}{6}$ for all i

$$b_i = \frac{1}{h}\left(f_{i+1} - 2f_i + f_{i-1}\right) \quad \text{for } i = 2, \ldots, n-2$$

$$b_1 = \frac{1}{h}\left(f_2 - 2f_1 + f_0\right) - \frac{h}{6} z_0$$

$$b_{n-1} = \frac{1}{h}\left(f_n - 2f_{n-1} + f_{n-2}\right) - \frac{h}{6} z_n$$

Note: We are free to choose z_0, z_n $\Rightarrow$ get different types of splines:

(1) $\boxed{z_0 = z_n = 0}$ $\left(S''(x_0) = S''(x_n) = 0\right)$ gives the natural cubic spline. (Rarely used)
Not accurate at x_0, x_n.

(2) $\underset{S''(x_0)}{z_0}$ & $\underset{S''(x_n)}{z_n}$ are chosen s.t. $\boxed{\begin{array}{l} S'(x_0) = f'(x_0) \\ S'(x_n) = f'(x_n) \end{array}}$

Called the complete cubic spline.
(need to have approx. $f'(x_0)$ and $f'(x_n)$)

(3) z_0 & z_n are chosen s.t. S''' is continuous at x_1 and x_{n-1}, i.e. $\boxed{\begin{array}{l} S_1'''(x_1) = S_2'''(x_1) \text{ and} \\ S_{n-2}'''(x_{n-1}) = S_{n-1}'''(x_{n-1}) \end{array}}$

(S_i is a piece of S on the ith subinterval)

Called $\underbrace{\text{not-a-knot}}$ spline (appropriate if $f'(x_0), f'(x_n)$ not known)

the idea is not to change

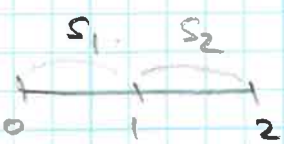
S as it crosses x_1 and x_{n-1} $\Rightarrow$ in this case, "they are not knots" : S''' being a constant and not-a-knot requirement imposes this!

Examples:

(1) Natural spline : $\underbrace{S''(x_0)}_{z_0} = \underbrace{S''(x_n)}_{z_n} = 0$.

Find it for $f(x) = x^4$ using subintervals $\underbrace{[0,1], [1,2]}_{n=2 \text{ intervals}}$. $\Rightarrow$ 3 knots: $x_0 = 0,\ x_1 = 1,\ x_2 = 2$
$f_0 = 0;\ f_1 = 1;\ f_2 = 16$

So, we use $z_0 = z_2 = 0$, let's find z_1 (only):



Using formulas: ($h = 1$)

$$d_1 = \frac{2h}{3} = 2/3 \qquad z_2 = 0$$

$$\beta_1 = \frac{h}{6} = 1/6 \quad \text{(don't even need this)}$$

$S_1''(0) = 0 = S_2''(2)$

$S_1(1) = S_2(1)$

$S_1'(1) = S_2'(1)$

$$b_1 = \frac{1}{h}(f_2 - 2f_1 - f_0) - \frac{h}{6} z_0$$
$$= (16 - 2 - 0) - 0 = 14$$

So, the system has one equation only:

$$d_1 z_1 = b_1 \quad \text{or} \quad \frac{2}{3} z_1 = 14 \Rightarrow z_1 = 21$$

$$\Rightarrow S_1(x) = 0 + \frac{1}{1} \cdot 21 \frac{(x-0)^3}{6} + \frac{1}{1}\left(1 - 0 + \frac{1^2}{6}(0-21)\right)(x-0)$$
$$+ 0 - \frac{1^2}{6} \cdot 0 = \frac{21x^3}{6} + \left(1 - \frac{21}{6}\right)x = \frac{7}{2}x^3 - \frac{5}{2}x$$

and
$$S_2(x) = 1 \cdot 1 \frac{(2-x)^3}{6} + 0 + \left[16 - 1 + \frac{1^2}{6}(21-0)\right](x-1)$$
$$+ 1 - \frac{1^2}{6} \cdot 21 = \frac{(2-x)^3}{6} + \left(16 - \frac{21}{6}\right)(x-1) = -\frac{7}{2}x^3 + 21x^2 - \frac{47}{2}x + 7$$

Thus, $s(x) = \begin{cases} \frac{7}{2}x^3 - \frac{5}{2}x, & 0 \le x \le 1 \\ -\frac{7}{2}x^3 + 21x^2 - \frac{47}{2}x + 7, & 1 \le x \le 2 \end{cases}$

(2) Determine the unknown parameters a, b, c, d, so that the function

$$S(x) = \begin{cases} \overbrace{x^2 + x^3}^{S_1(x)}, & [0,1] \\ \underbrace{a + bx + cx^2 + dx^3}_{S_2(x)}, & [1,2] \end{cases}$$

is a cubic spline w/ $S_2'''(x) = 12$.

We should have:

$$S_1(1) = S_2(1), \quad S_1'(1) = S_2'(1), \quad S_2''(1) = S_2''(1)$$

and $S_2'''(x) = 12$

$$S_1'(x) = 2x + 3x^2, \quad S_2'(x) = b + 2cx + 3dx^2$$

$$S_1''(x) = 2 + 6x, \quad S_2''(x) = 2c + 6dx, \quad S_2''' = 6d = 12$$

$$\Downarrow$$

$$\boxed{d = 2}$$

$$5 = b + 2c + 3d$$

$$8 = 2c + 6d$$

Since $d = 2 \Rightarrow c = -2 \Rightarrow 5 = b - 4 + 6 \Rightarrow b = 3$

From $S_1(1) = S_2(1) \Rightarrow 2 = a + b + c + d \Rightarrow a = -1$

So, $S_2(x) = -1 + 3x - 2x^2 + 2x^3$ on $[1,2]$.

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